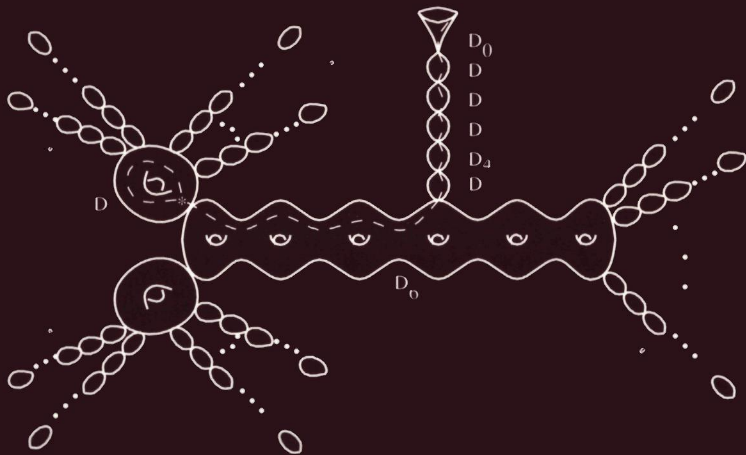
On De Rham Homotopy Theory for Plane Algebraic Curves and their Singularities



Rainer Kaenders

On De Rham Homotopy Theory for Plane Algebraic Curves and their Singularities

een wetenschappelijke proeve op het gebied van de
Wiskunde en Informatica

Proefschrift

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für Claudia

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Contents

Introduction

The building block for many invariants of plane curve singularities like the monodromy, the intersection form, the Seifert form, Coxeter-Dynkin diagrams, the mixed Hodge structure on the vanishing cohomology and the spectral pairs is the first integral homology group of the Milnor fiber or – which is the same – the abelianized fundamental group of the Milnor fiber.

The restriction to the abelianized version of this fundamental group causes some serious limitations. E. g. Lê Dũng Tráng [Lê72] proved that for irreducible plane curve singularities the monodromy (on homology) has finite order. However A'Campo [A'C73] was the first to discover that the monodromy on the fundamental group of the Milnor fiber (where the one boundary component is collapsed to one point, the basepoint) has infinite order, whenever the singularity has at least two Puiseux pairs. This shows that the monodromy on the fundamental group is richer than on homology.

For the study of analytical invariants of branches of plane curve singularities, the first (co)homology group is of particular interest because of Hodge theory.

Deligne [Del71] introduced in 1970/71 the notion of mixed Hodge structure (MHS) and showed that the cohomology of any complex algebraic variety carries such a MHS in a natural way. Schmid [Sch73] and Steenbrink [Ste76] determined a limit-MHS of a variation of Hodge structures coming from families of compact algebraic manifolds. This provided the background for the definition of the MHS on the vanishing cohomology of hypersurface singularities by Steenbrink [Ste77]. In the case of plane curve singularities this is a MHS on the first cohomology group of the Milnor fiber.

However, due to the fact that for irreducible plane curve singularities the monodromy on homology is finite, families of mutually right-left – but not necessarily right-equivalent plane curve singularities all have isomorphic MHSs on the vanishing cohomology. This holds, for example, for the family

$$f_\lambda(x,y) = \left(y^2 - x^3\right)^2 - \left(4\lambda x^5 y + \lambda^2 x^7\right), \quad \lambda \neq 0, \tag{0.1}$$

which is the singularity with Puiseux expansion

$$y = x^{\frac{3}{2}} + \sqrt{\lambda}\, x^{\frac{7}{4}}.$$

In 1987 Hain [Hai87a], [Hai87c] used Chen's iterated integrals to generalize Deligne's construction of a MHS on cohomology to a construction of a MHS on the homotopy Lie algebra for any complex algebraic variety with base point. In particular, this object contains a MHS which is called *the MHS on the fundamental group*. Already before, in 1978, Morgan [Mor78] had used Sullivan's minimal models to put a MHS on the homotopy Lie algebra of *smooth* complex algebraic varieties.

Hain also showed, in [Hai87b], that the local system of homotopy groups associated with a topologically trivial family of smooth pointed varieties underlies a good variation of $\mathbb{Q}$-MHSs, which implies by the earlier work of Steenbrink and Zucker [SZ85] that there is a limit $\mathbb{Q}$-MHS.

This provides the background of the problem that Steenbrink proposed to us: Is there a generalization of the MHS on the vanishing (co)homology to something like 'a MHS on the fundamental group of the general Milnor fiber' in the case of (irreducible) plane curve singularities?

In this thesis we show that the construction of such a generalization is possible and we will call it *the mixed Hodge structure on the nearby fundamental group of an irreducible plane curve singularity*. We develop a theory that allows us to take iterated integrals along certain paths in the central fiber. These paths mimic paths in the regular fiber. The given approach seems to be even new for the integration of elements in the vanishing cohomology. Due to this integration theory, the constructed MHS is defined over the integers. For instance in the family f_λ defined on page 7 this $\mathbb{Z}$-MHS detects the modulus.

The above formulated problem is the guideline throughout this whole thesis. However, the construction of the mixed Hodge structure on the nearby fundamental group is subject to Part II.

Part I arose from a consideration in the preparation for this construction. Since the Milnor fiber is a non compact Riemann surface, it was our first goal to understand Hain's MHS on the fundamental group in a *very simple case of a non-compact Riemann surface*: the complement of one point in a compact Riemann surface, which we refer to as *a punctured compact Riemann surface*.

In the case of compact Riemann surfaces Hain and Pulte ([Hai87c], [Pul88]) proved that the extension of MHSs, which is given by the weights 1 and 2 of the MHS on the fundamental group, determines the base point (see Theorem 4.1 on page 57). From this result and from the classical Torelli theorem they derived *a pointed Torelli theorem* as a corollary (see 4.3 on page 58).

For a punctured Riemann surface $(X \setminus \{q\}, p)$, the extension that is given by the weights 1 and 2, is one dimension bigger than in the compact case. Call this extension m_{pq}. It determines in a natural way on the one hand the corresponding extension of (X, p) and on the other hand an element in the Picard group $\mathrm{Pic}^0 X$. We prove that this element in $\mathrm{Pic}^0 X$ is given by

$$(2g\, q - 2p - K),$$

where K is the canonical divisor and g the genus of X (see Theorem 3.4 on page 42). Furthermore, we show that this extension m_{pq} determines both, the base point p and the 'removed point' q. This, together with the pointed Torelli theorem of Hain and Pulte, yields a *two-pointed Torelli theorem* as a corollary (see 4.8 on page 59).

In Part II we consider first the situation of a degenerating family $h : (Z, D^+) \to (\Delta, 0)$ of compact Riemann surfaces with one boundary component over the disk $\Delta \subset \mathbb{C}$, where the singular fiber, the fiber over 0, is a divisor with normal crossings $D = D^+ \cup D_0$. Its components are compact closed Riemann surfaces, except for the one component D_0, which is a disk. Such a degenerating family can be constructed from an irreducible plane curve singularity by a process that is called 'semistable reduction' (see 8.1).

In this situation, we want to mimic a path in a regular fiber by a path in the central fiber. Let us give a finite version of the ideas that we develop in an infinitesimal way in Chapter 5. Intuitively it is clear that the difficulties for this mimicry arise at the double points of the divisor with normal crossings D, which are all given locally by an equation: $x \cdot y = t$. If $t \neq 0$, this equation gives (locally) a 1-1 relation between points off the double point in one component and points off the double point in the other component. This is obviously not true for $t = 0$. For $t \neq 0$, we consider then paths γ that approach the double point in one component and leave it in the other component in such a way that locally for all $\varepsilon > 0$ holds:

$$x \circ \gamma(\tau_0 - \varepsilon) \cdot y \circ \gamma(\tau_0 + \varepsilon) = t,$$

where $\gamma(\tau_0)$ is the double point. By the identification that is sketched in Figure 0.1, these paths can be thought of as being in the fiber over t.

In this picture one can also observe that if $|t|$ is big, the path in the central fiber has to make a big detour compared to the corresponding path in the regular fiber. Therefore, the speed with which the path in the central fiber approaches and leaves the double point is a measure for the distance between the central fiber and the regular fiber in which the 'regular path' lives. We give an infinitesimal version of this idea in 5.3. Given a non zero tangent vector $\vec{v}$ in the tangent plane $T_0\Delta$ of the base space Δ and a non zero tangent vector $\vec{w}$ in the tangent plane $T_{p_0}D_0$ at the disk D_0, we introduce the notion of *a path over $\vec{v}$ based at $\vec{w}$*. Moreover, we define a *homotopy over $\vec{v}$ based at $\vec{w}$* or shorter, a *nearby homotopy*, in such a way that the set of all *nearby homotopy classes* becomes a group that we denote by

$$\pi_1(Z_{\vec{v}}, \vec{w}).$$

The $\{\pi_1(Z_{\vec{v}}, \vec{w})\}_{\vec{v} \in (T_0\Delta)^*, \vec{w} \in (T_{p_0}D_0)^*}$ form a local system on $(T_0\Delta)^* \times (T_{p_0}D_0)^*$, which is isomorphic to a corresponding local system that comes from the fundamental groups on the regular fibers (see 5.8 on page 85).

Denote by $\vec{J} = J_{\vec{v},\vec{w}}$ the augmentation ideal of the group ring $\mathbb{Z}\pi_1(Z_{\vec{v}}, \vec{w})$. In the Chapters 6 and 7, we will define for all $s \geq 0$ a $\mathbb{Z}$-MHS on

$$\mathbb{Z}\pi_1(Z_{\vec{v}}, \vec{w}) / J^{s+1}_{\vec{v},\vec{w}}. \tag{0.2}$$

This will be done by putting a $\mathbb{Z}$-MHS on $(\vec{J}/\vec{J}^{s+1})^*$, the dual of $\vec{J}/\vec{J}^{s+1} \subset \mathbb{Z}\pi_1(Z_{\vec{v}}, \vec{w}) / J^{s+1}_{\vec{v},\vec{w}}$.

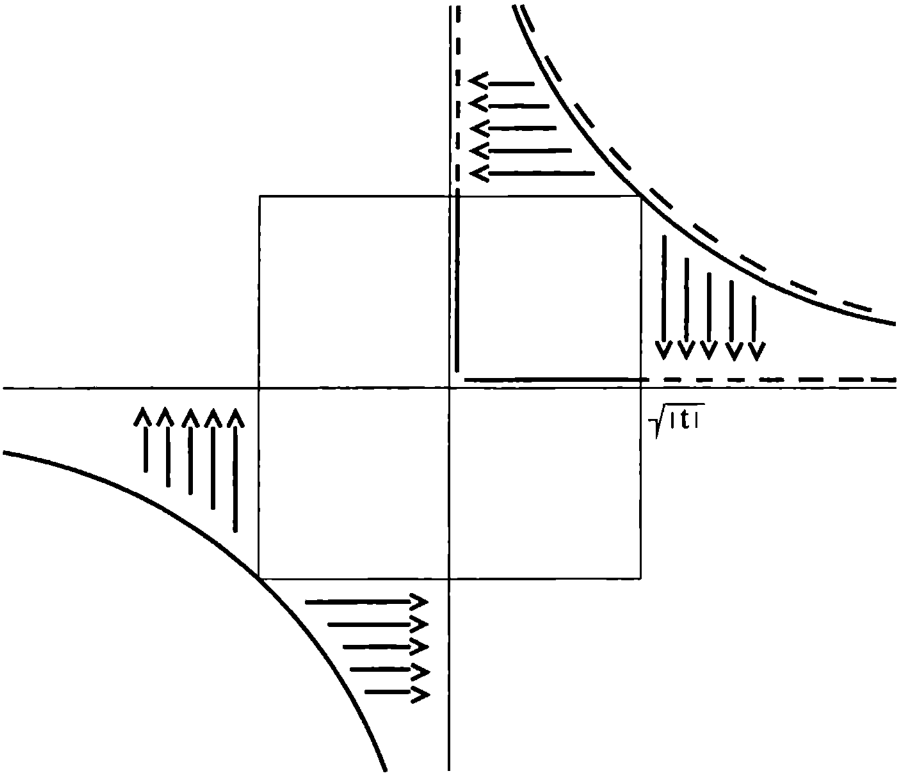

Figure 0.1: Identification of a path in the regular fiber of the map $(x, y) \mapsto x \cdot y = t$ with a path in the singular fiber.

The main ingredient for the definition of these $\mathbb{Z}$-MHSs is a differential graded algebra (dga) $A^\bullet$, which is made up from differential forms on the central fiber but whose cohomology is isomorphic to the cohomology of a regular fiber. It was suggested to us by Hain[2] to use a dga like $A^\bullet$ and he proposed the problem of finding a way to integrate elements of this dga along some kind of paths in the central fiber. We discovered that the right notion of *path* for this purpose turned out to be the notion of a path over $\vec{v}$ (based at $\vec{w}$). We will show that closed elements in A^1 can be integrated along paths in Z_0 over $\vec{v}$ in a natural way. This yields an integral structure on $H^1(A^\bullet)$, which is by this integration dual to $\vec{J}/\vec{J}^2$. On $A^\bullet$ we have filtrations $W_\bullet$ and $F^\bullet$ such that the induced filtrations together with the integral structure define a $\mathbb{Z}$-MHS on $H^1(A^\bullet)$ (see Theorem 6.11 on page 101).

Since $A^\bullet$ has an *algebra* structure, we can also define *iterated* line integrals along paths over $\vec{v}$. We shall do this in Chapter 7. This will give a way to

[2]The dga $A^\bullet$ is in principle a modified version of the complex $P^\bullet_{\mathbb{C}}(X^*)[\theta]$ of [Hai87b] for this special situation with non compact fibers.

describe $\mathrm{Hom}_{\mathbb{Z}}(\vec{J}/\vec{J}^{s+1}, \mathbb{C})$ in terms of the dga $A^\bullet$, similar to Chen's classical π_1-De-Rham theorem (see Theorem 1.15 on page 23 in Chapter 1 and Theorem 7.17 on page 136 in Chapter 7).

These iterated integrals will lead to the definition of a $\mathbb{Z}$-MHS on $(\vec{J}/\vec{J}^{s+1})^*$, and hence to a $\mathbb{Z}$-MHS on (0.2) (see Theorem 7.20 on page 139). And furthermore, we will see that the variation of $\mathbb{Z}$-MHS for all $s \geq 1$

$$\left\{J_{\vec{v},\vec{w}}/J^{s+1}_{\vec{v},\vec{w}}\right\}_{\vec{v}\in(T_0\Delta)^\bullet,\ \vec{w}\in(T_{p_0}D_0)^\bullet}$$

is a nilpotent orbit of MHS (see 6.4 and 7.5).

Let $\pi_1(Z_0, p_0)$ be the fundamental group of the central fiber and let $\mathbb{Z}\pi_1(Z_0, p_0)$ be its group ring with augmentation ideal J_0. We define a MHS on the fundamental group of the central fiber. This MHS is more or less a special case of the general construction of a MHS on the fundamental group of a complex algebraic varietiy as introduced by Hain [Hai87a]. We show that the obvious group homomorphism

$$c : \pi_1(Z_{\vec{v}}, \vec{w}) \to \pi_1(Z_0, p_0)$$

induces an inclusion of MHSs:

$$c^* : \left(J_0/J_0^{s+1}\right)^* \to \left(\vec{J}/\vec{J}^{s+1}\right)^*.$$

In Chapter 8 we show how all these considerations can be applied to study irreducible plane curve singularities

$$f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0).$$

The tangent vector $\frac{\partial}{\partial t}$ in $T_0\mathbb{C}$ yields a finite number of tangent vectors in the base space of the 'semistable reduction'. We show that it also defines a finite number of tangent vectors in $T_{p_0}D_0$ in a natural way, which we call *the monstrance of f*. Finally we consider the *MHS on the nearby fundamental group* of the example (0.1) on page 7 and show that the extension of H_1 by the part of weight -3 of $\vec{J}/\vec{J}^4$ detects the modulus that is hidden for the vanishing cohomology. Here we focus on the effects due to the infinite order of the monodromy.

In general we expect also interesting information from other parts of this MHS on the nearby fundamental group. Even the extension of H_1 by $H_1 \otimes H_1$ given by $\vec{J}/\vec{J}^3$ might contain interesting information. Also the decomposition of the monodromy into a unipotent and a semisimple part has still to be studied. We think that the ideas described here for irreducible plane curve singularities will have a generalization for reducible plane curve singularities. We know that they will have a counterpart for degenerating families of compact Riemann surfaces with a section (representing the basepoint in each fiber).

Part I

Mixed Hodge Structure on the Fundamental Group of a Punctured Riemann Surface

Chapter 1

Iterated Integrals and Chen's De Rham theorem

In this section we introduce iterated line integrals on smooth real or complex manifolds and show how they are related with the fundamental group. We choose an approach different from [Che77b] and [Hai87c], which does not need iterated integrals over forms of degree higher than 1. The main idea of our approach is to express iterated line integrals on a manifold as ordinary line integrals on the universal covering space of this manifold.

We introduce *Chen's differential* and find that the algebraic way to say that an iterated integral is a homotopy functional is to say that Chen's differential vanishes on this iterated integral.

Consider the augmentation ideal of the group ring of the fundamental group truncated by one of its powers. We will discuss Chen's π_1-De-Rham theorem, which establishes an isomorphism between the dual of this object and a certain vector space of functions on the fundamental group coming from iterated integrals.

1.1 Definition and Elementary Properties

Let M be a smooth connected manifold over $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Denote by PM the set of piecewise smooth paths $\underline{\gamma} : [0;1] \longrightarrow M$.

Definition 1.1 An *iterated line integral of length* $\leq s$ *with values in* $\mathbb{K}$ is by definition a linear combination of functions from PM to $\mathbb{K}$, which are either constant or of the following type:

Let $\varphi_1, \ldots \varphi_r$ with $0 \leq r \leq s$ be smooth $\mathbb{K}$-valued 1-forms on M. We denote by $\int \varphi_1 \cdots \varphi_r$ the function, which maps $\underline{\gamma} \in PM$ to the complex number,

$$\int_{\underline{\gamma}} \varphi_1 \cdots \varphi_r := \int_0^1 \int_0^{t_r} \cdots \int_0^{t_2} f_1(t_1) \cdots f_r(t_r) dt_1 \cdots dt_r \quad ,$$

where $f_j(t)dt$ is defined to be $\underline{\gamma}^* \varphi_j$, $j = 1, \ldots, r$.

If $f : M \longrightarrow N$ is a smooth map and $\int I$ an iterated integral on N, then $\int f^*I$ is defined in the obvious way by pulling back the φ_j's by f. One finds immediately that for any $\underline{\gamma} \in PM$ holds:

$$\int_{f\circ\underline{\gamma}} I = \int_{\underline{\gamma}} f^*I \,.$$

Remark 1.2 Since we only work with iterated line integrals in this thesis we will call them simply *iterated integrals.*

There are several very useful and easy to verify properties of iterated integrals such as the following.

Proposition 1.3 *Let $\underline{\gamma}$, $\underline{\alpha}$ and $\underline{\beta}$ be be paths in PM and and let $\varphi_1, \ldots, \varphi_r$ be smooth 1-forms on M. Moreover, let h be a smooth function on M. Then the following equalities hold:*

- $\int\limits_{\underline{\gamma}} dh\varphi_2 \cdots \varphi_r = \int\limits_{\underline{\gamma}} (h\varphi_2)\varphi_3 \cdots \varphi_r - h(\underline{\gamma}(0)) \int\limits_{\underline{\gamma}} \varphi_2 \cdots \varphi_r;$
- $\int\limits_{\underline{\gamma}} \varphi_1 \cdots \varphi_{i-1} dh \varphi_{i+1} \cdots \varphi_r$
  $= \int\limits_{\underline{\gamma}} \varphi_1 \cdots \varphi_{i-1}(h\varphi_{i+1})\varphi_{i+2} \cdots \varphi_r - \int\limits_{\underline{\gamma}} \varphi_1 \cdots \varphi_{i-2}(h\varphi_{i-1})\varphi_{i+1} \cdots \varphi_r;$
- $\int\limits_{\underline{\gamma}} \varphi_1 \cdots \varphi_{r-1} dh = h(\underline{\gamma}(1)) \int\limits_{\underline{\gamma}} \varphi_1 \cdots \varphi_{r-1} - \int\limits_{\underline{\gamma}} \varphi_1 \cdots \varphi_{r-2}(h\varphi_{r-1})$

and also

- $\int\limits_{\underline{\gamma}} \varphi_1 \cdots \varphi_r = (-1)^r \int\limits_{\underline{\gamma}^{-1}} \varphi_r \cdots \varphi_1$

as well as

- $\int\limits_{\underline{\alpha}\star\underline{\beta}} \varphi_1 \cdots \varphi_r = \sum\limits_{0 \leq m \leq r} \int\limits_{\underline{\alpha}} \varphi_1 \cdots \varphi_m \int\limits_{\underline{\beta}} \varphi_{m+1} \cdots \varphi_r.$ □

Let p and q be two points of M. As a consequence of 1.3 we see that if one of the φ_ν is exact, then there is an iterated integral of lower length, which yields the same function as the original iterated integral, when restricted to $P(M; p, q)$, the set of all paths in PM starting in p and ending in q.

Denote for a point $p \in M$ by $P(M,p)$ the set of all paths in PM with basepoint p (that is $P(M,p) := P(M;p,p)$). Observe that the restriction of iterated integrals $\int I : PM \longrightarrow \mathbb{K}$ to functions $\int I : P(M,p) \longrightarrow \mathbb{K}$ is in general not injective. E. g. $\int dz : P(\mathbb{C}, 0) \to \mathbb{C}$ is the zero function. We can consider an iterated integral, restricted to $P(M,p)$, also as a function on $\mathbb{K}P(M,p)$, the free abelian group generated by the elements of $P(M,p)$, by extending it linearly.

Proposition 1.3 allows us to consider an iterated integral of closed forms of length r as an iterated integral on the universal covering space of length $r-1$.

For example consider $\underline{\gamma} \in PM$ and two closed 1-forms φ, ψ on M with their liftings $\tilde{\varphi}$, $\tilde{\psi}$ on the universal covering space $\pi : \tilde{M} \longrightarrow M$. Let $\tilde{\underline{\gamma}} \in P\tilde{M}$ be a lifting of $\underline{\gamma}$. Since on $\tilde{M}$ every closed form is exact, there is a function h on $\tilde{M}$ with $h(\tilde{\underline{\gamma}}(0)) = 0$ such that $\tilde{\varphi} = dh$. Then

$$\int_{\underline{\gamma}} \varphi\psi = \int_{\tilde{\underline{\gamma}}} \tilde{\varphi}\tilde{\psi} = \int_{\tilde{\underline{\gamma}}} h\tilde{\psi}$$

and $h\tilde{\psi}$ does not need to be closed ($h\tilde{\psi}$ closed $\Leftrightarrow \varphi \wedge \psi = 0$) in general, although φ and ψ are closed. As a consequence, an iterated integral does not take the same value on homotopic paths in general.

Definition 1.4 We say that an iterated integral is a *homotopy functional*, if it only depends on the homotopy class of a path relative to its endpoints. After restriction to $P(M,p)$, such an iterated integral can not only be considered as function on $\mathbb{K}P(M,p)$ but also on the group ring $\mathbb{K}\pi_1(M,p)$.

If we consider closed 1-forms φ_i, ψ_i and a 1-form μ then

$$\int \sum \varphi_i \psi_i + \mu \tag{1.1}$$

is a homotopy functional if and only if $\sum \varphi_i \wedge \psi_i + d\mu = 0$, as one can see by using the preceding to write it as an ordinary integral of a closed (!) 1-form on $\tilde{M}$. In this way we find the properties $d\varphi_i = 0$, $d\psi_i = 0$ and $\sum \varphi_i \wedge \psi_i + d\mu = 0$ to be a criterion to ensure that the iterated integral (1.1) of length ≤ 2 is a homotopy functional. This observation will be generalized in Theorem 1.13.

Also the next proposition is important for understanding iterated integrals as functionals on paths.

Proposition 1.5 *Let $\underline{\alpha}_1, \dots, \underline{\alpha}_s \in P(M,p)$ and $\varphi_1, \dots, \varphi_r$ be smooth $\mathbb{K}$-valued 1-forms on M, then for $s \geq r$:*

$$\int_{(\underline{\alpha}_1 - 1)\cdots(\underline{\alpha}_s - 1)} \varphi_1 \cdots \varphi_r = \begin{cases} \int_{\underline{\alpha}_1} \varphi_1 \cdots \int_{\underline{\alpha}_s} \varphi_s & , \text{ if } r = s \\ 0 & , \text{ if } r < s. \end{cases}$$

Proof: Define $\Delta(I^s) := \{(t_1, \dots, t_s) \in I^r \mid 0 \leq t_1 \leq \cdots \leq t_s \leq 1\}$ and $I_i := [\frac{i-1}{s}; \frac{i}{s}]$. For closed paths, based at p, $\underline{\alpha}_i : I_i \to M$, $i = 1, \dots, s$ and $\underline{\gamma} := \underline{\alpha}_1 * \cdots * \underline{\alpha}_s$ and functions $f_1, \dots, f_r$ with $f_j(t)dt = \underline{\gamma}^* \varphi_j$, $j = 1, \dots, r$ it is very easy to prove the following formula by induction on k:

$$\int_{(\underline{\alpha}_1 - 1)\cdots(\underline{\alpha}_k - 1)\underline{\alpha}_{k+1}\cdots\underline{\alpha}_s} \varphi_1 \cdots \varphi_r = \sum_{\substack{\{i_1, \dots, i_r\} \supseteq \{1, \dots, k\} \\ 1 \leq i_1 \leq \cdots \leq i_r \leq s}} \int_{I_{i_1} \times \cdots \times I_{i_r} \cap \Delta(I^r)} f_1(t_1) \cdots f_r(t_r) dt_1 \cdots dt_r.$$

For $k > r$ the sum is zero and for $k = s = r$ there is only one summand. The proof is completed by applying Fubini. $\square$

1.2 Review on Differential Graded Algebras

In this subsection we want to describe some properties of differential graded algebras (dga's). Let $\mathcal{A}^\bullet$ be a connected dga over $\mathbb{K}$ with augmentation map $\epsilon : \mathcal{A}^\bullet \to \mathbb{K}$ and let $\mathcal{R}(\mathcal{A}^\bullet, \epsilon)$ be the sub-vector space of

$$\bigoplus_{r=0}^{\infty} \bigotimes^r \mathcal{A}^\bullet$$

generated by elements $R_i(u, f)$, $i = 1, \ldots, r$, which are defined for $f \in \mathcal{A}^0$ and $u = \varphi_1 \otimes \cdots \otimes \varphi_r \in \bigotimes^r \mathcal{A}^1$ by:

$$R_1(u,f) := df \otimes \varphi_2 \otimes \cdots \otimes \varphi_r - (f - \epsilon(f))\varphi_2 \otimes \varphi_3 \otimes \cdots \otimes \varphi_r, \tag{1.2}$$

$$\begin{aligned} R_i(u,f) :=& \varphi_1 \otimes \cdots \otimes \varphi_{i-1} \otimes df \otimes \varphi_{i+1} \otimes \cdots \otimes \varphi_r \\ &+ \varphi_1 \otimes \cdots \otimes f\varphi_{i-1} \otimes \varphi_{i+1} \otimes \cdots \otimes \varphi_r \\ &- \varphi_1 \otimes \cdots \otimes \varphi_{i-1} \otimes f\varphi_{i+1} \otimes \cdots \otimes \varphi_r \quad \text{for } 1 < i < r, \end{aligned} \tag{1.3}$$

$$R_r(u,f) := \varphi_1 \otimes \cdots \otimes \varphi_{r-1} \otimes df + \varphi_1 \otimes \cdots \otimes (f - \epsilon(f))\varphi_{r-1}. \tag{1.4}$$

We define two maps[1], the *internal* and the *combinatorial differential*,

$$d_I,\ d_C : \bigoplus_{r=0}^{\infty} \bigotimes^r \mathcal{A}^1 \longrightarrow \bigoplus_{r=0}^{\infty} \bigotimes^r \mathcal{A}^\bullet$$

by giving their values on elements of the form $\varphi_1 \otimes \cdots \otimes \varphi_s$ and extending $\mathbb{K}$-linearly:

$$\begin{aligned} d_I(\varphi_1 \otimes \cdots \otimes \varphi_s) &:= \sum_{i=1}^{s} \varphi_1 \otimes \cdots \otimes \varphi_{i-1} \otimes d\varphi_i \otimes \varphi_{i+1} \otimes \cdots \otimes \varphi_s\,, \\ d_C(\varphi_1 \otimes \cdots \otimes \varphi_s) &:= \sum_{i=1}^{s-1} \varphi_1 \otimes \cdots \otimes \varphi_{i-1} \otimes \varphi_i \wedge \varphi_{i+1} \otimes \varphi_{i+2} \otimes \cdots \otimes \varphi_s. \end{aligned}$$

Here the maps d_I and d_C shift the degrees in the following way:

$$d_I : \bigotimes^r \mathcal{A}^1 \longrightarrow \left(\bigotimes^r [\mathcal{A}^1 \oplus \mathcal{A}^2]\right)^{r+1}, \quad d_C : \bigotimes^r \mathcal{A}^1 \longrightarrow \left(\bigotimes^{r-1} [\mathcal{A}^1 \oplus \mathcal{A}^2]\right)^r.$$

Definition 1.6 The sum $(d_I + d_C)$ induces a map

$$d_{\mathrm{Chen}} : \bigoplus_{r=0}^{\infty} \bigotimes^r \mathcal{A}^1 \longrightarrow \left(\bigoplus_{r=0}^{\infty} \bigotimes^r \mathcal{A}^\bullet\right) \Big/ \mathcal{R}(\mathcal{A}^\bullet, \epsilon),$$

which we call *Chen differential* and its kernel consists of what we call *Chen-closed elements.*

[1]Define $\bigotimes^0 \mathcal{A}^\bullet$ to be $\mathbb{K}$ and $d_C : \mathcal{A}^1 \to \mathbb{K}$ to be the zero map.

For each s the kernel of d_{Chen} gets a name

$$\mathcal{K}^s(\mathcal{A}^\bullet,\epsilon) := \ker\left\{ d_{\mathrm{Chen}} : \bigoplus_{r=0}^{s} \bigotimes^r \mathcal{A}^1 \to \left(\bigoplus_{r=0}^{\infty} \bigotimes^r \mathcal{A}^\bullet\right) \Big/ \mathcal{R}(\mathcal{A}^\bullet,\epsilon) \right\}$$

and $\bar{\mathcal{K}}^s(\mathcal{A}^\bullet,\epsilon) := \mathcal{K}^s(\mathcal{A}^\bullet,\epsilon) \cap \bigoplus_{r=1}^{s} \bigotimes^r \mathcal{A}^1$, i. e. $\bar{\mathcal{K}}^s(\mathcal{A}^\bullet,\epsilon)$ are the elements of $\mathcal{K}^s(\mathcal{A}^\bullet,\epsilon)$ with zero constant term. Define moreover

$$\mathcal{R}_1^s(\mathcal{A}^\bullet,\epsilon) := \left(\bigoplus\nolimits_{r=0}^{s} \bigotimes^r \mathcal{A}^1\right) \cap \mathcal{R}(\mathcal{A}^\bullet,\epsilon)$$

and

$$\bar{\mathcal{R}}_1^s(\mathcal{A}^\bullet,\epsilon) := \left(\bigoplus\nolimits_{r=1}^{s} \bigotimes^r \mathcal{A}^1\right) \cap \mathcal{R}(\mathcal{A}^\bullet,\epsilon)$$

and notice that $\mathcal{R}_1^s(\mathcal{A}^\bullet,\epsilon) \subset \mathcal{K}^s(\mathcal{A}^\bullet,\epsilon)$ and $\bar{\mathcal{R}}_1^s(\mathcal{A}^\bullet,\epsilon) \subset \bar{\mathcal{K}}^s(\mathcal{A}^\bullet,\epsilon)$.

Definition 1.7

$H^0B_s(\mathcal{A}^\bullet,\epsilon) := \mathcal{K}^s(\mathcal{A}^\bullet,\epsilon)/\mathcal{R}_1^s(\mathcal{A}^\bullet,\epsilon)$ and $H^0\bar{B}_s(\mathcal{A}^\bullet,\epsilon) := \bar{\mathcal{K}}^s(\mathcal{A}^\bullet,\epsilon)/\bar{\mathcal{R}}_1^s(\mathcal{A}^\bullet,\epsilon)$.

An element of $H^0B_s(\mathcal{A}^\bullet,\epsilon)$ respectively $H^0\bar{B}_s(\mathcal{A}^\bullet,\epsilon)$ that is represented by $\sum_J a_J\, \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}$ is classically notated by $\sum_J a_J\, (\varphi_{j_1}|\cdots|\varphi_{j_r})$.

Observe that we have for all $s \geq 1$ inclusions

$$H^0B_s(\mathcal{A}^\bullet,\epsilon) \subset H^0B_{s+1}(\mathcal{A}^\bullet,\epsilon) \quad \text{and} \quad H^0\bar{B}_s(\mathcal{A}^\bullet,\epsilon) \subset H^0\bar{B}_{s+1}(\mathcal{A}^\bullet,\epsilon).$$

Remark 1.8 A more general version of this construction of the Chen differential d_{Chen} and of the vectorspaces $H^0B_s(\mathcal{A}^\bullet,\epsilon)$ and $H^0\bar{B}_s(\mathcal{A}^\bullet,\epsilon)$ is *the reduced bar construction*, which is described in ample generality in [Che76] or [Hai87a].

Let $\bar{\mathcal{A}}^\bullet$ be a connected sub-dga of $\mathcal{A}^\bullet$ such that $\bar{\mathcal{A}}^p = \mathcal{A}^p$ for $p > 1$ and

$$\mathcal{A}^1 = d\mathcal{A}^0 \oplus \bar{\mathcal{A}}^1. \tag{1.5}$$

The proof of the following two lemmas is similar to the proof of the Theorem on p.23 in [Che76].

Lemma 1.9 *Suppose $I \in \bigoplus_{r=0}^{s} \bigotimes^r \mathcal{A}^1$. Then there is an $R \in \mathcal{R}_1^s(\mathcal{A}^\bullet,\epsilon)$ and an $\bar{I} \in \bigoplus_{r=0}^{s} \bigotimes^r \bar{\mathcal{A}}^1$ such that $I = \bar{I} + R$ and*

$$d_{Chen} I = 0 \quad \Leftrightarrow \quad (d_I + d_C)(\bar{I}) = 0.$$

Proof: By finite induction on $\bar{s}$ we prove that for every $\bar{s}$ between s and 0 there is an $I^{\bar{s}} \in \bigoplus_{r=0}^{\bar{s}} \bigotimes^r \mathcal{A}^1$ and an $R_{\bar{s}} \in \mathcal{R}_1^s(\mathcal{A}^\bullet,\epsilon)$ as well as an $\bar{I}_{\bar{s}} \in \bigoplus_{r=0}^{s} \bigotimes^r \bar{\mathcal{A}}^1$ such that $I = \bar{I}_{\bar{s}} + I^{\bar{s}} + R_{\bar{s}}$.

First, for $\bar{\mathbf{s}} = \mathbf{s}$ let $I^{\bar{s}} = I$ and $\bar{I}_{\bar{s}} = R_{\bar{s}} = 0$.

$\tilde{s} < s$: Assume now that there is an $I^{\tilde{s}+1} \in \bigoplus_{r=0}^{\tilde{s}+1} \bigotimes^r \mathcal{A}^1$ and an $R_{\tilde{s}+1} \in \mathcal{R}_1^s(\mathcal{A}^\bullet, \epsilon)$ as well as an $\bar{I}_{\tilde{s}+1} \in \bigoplus_{r=0}^{s} \bigotimes^r \bar{\mathcal{A}}^1$ such that

$$I = \bar{I}_{\tilde{s}+1} + I^{\tilde{s}+1} + R_{\tilde{s}+1}.$$

Write $I^{\tilde{s}+1} = \sum_{|J|\le \tilde{s}+1} a_J\ \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}$ and let $\varphi_{j_\nu} = \bar{\varphi}_{j_\nu} + df_{j_\nu}$ be the decomposition according to (1.5). Then observe that

$$\begin{aligned} I^{\tilde{s}} := \sum_{|J|\le \tilde{s}+1} a_J \quad \{ & df_{j_1} \otimes \varphi_{j_2} \otimes \cdots \otimes \varphi_{j_r} && - R_1(\varphi_{j_2} \otimes \cdots \otimes \varphi_{j_r}; f_{j_1}) \\ & + \bar{\varphi}_{j_1} \otimes df_{j_2} \otimes \cdots \otimes \varphi_{j_r} && - R_2(\bar{\varphi}_{j_1} \otimes \varphi_{j_3} \otimes \cdots \otimes \varphi_{j_r}; f_{j_2}) \\ & \vdots && \vdots \\ & + \bar{\varphi}_{j_1} \otimes \cdots \otimes \bar{\varphi}_{j_{r-1}} df_{j_r} && - R_r(\bar{\varphi}_{j_1} \otimes \cdots \otimes \bar{\varphi}_{j_{r-1}}; f_{j_r}) \} \end{aligned}$$

and $\bar{I}_{\tilde{s}} := \sum_{|J|\le \tilde{s}+1} a_J\ \bar{\varphi}_{j_1} \otimes \cdots \otimes \bar{\varphi}_{j_r}$ as well as

$$\begin{aligned} R_{\tilde{s}} := \sum_{|J|\le \tilde{s}+1} a_J \{ \quad & R_1(\varphi_{j_2} \otimes \cdots \otimes \varphi_{j_r}; f_{j_1}) \\ + \quad & R_2(\bar{\varphi}_{j_1} \otimes \varphi_{j_3} \otimes \cdots \otimes \varphi_{j_r}; f_{j_2}) \\ & \vdots \\ + \quad & R_r(\bar{\varphi}_{j_1} \otimes \cdots \otimes \bar{\varphi}_{j_{r-1}}; f_{j_r}) \} \end{aligned}$$

satisfy the induction hypothesis for $\tilde{s}$. This completes the induction. Finally let $\bar{I} := \bar{I}_0$ and note that $d_{\mathrm{Chen}}\bar{I} = 0 \Leftrightarrow (d_I + d_C)\bar{I} = 0$. □

The following observation is a very useful lemma.

Lemma 1.10 *Suppose $I \in \bigoplus_{r=0}^{s} \bigotimes^r A^1$ is Chen-closed. Then we can write*

$$I = \sum_{|J|=r\le s} a_J\ \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r},$$

where for any J with $|J| = s$ and $a_J \neq 0$ either all $\varphi_{j_1}, \ldots, \varphi_{j_s}$ are closed or at least one of them is exact.

Proof: Let $I \in \bigoplus_{r=0}^{s} \bigotimes^r \mathcal{A}^1$ be Chen-closed. By Lemma 1.9 we may assume without loss of generality that $I \in \bigoplus_{r=0}^{s} \bigotimes^r \bar{\mathcal{A}}^1$ and $(d_I + d_C)I = 0$. Write $I = \sum_J a'_J\ \varphi'_{j_1} \otimes \cdots \otimes \varphi'_{j_r}$ and consider the part of I, which is contained in the s-th tensor power of $\bar{\mathcal{A}}^1$, say $T^s := \sum_{|J|=s} a'_J\ \varphi'_{j_1} \otimes \cdots \otimes \varphi'_{j_r}$, as element of $[\bigotimes^s \mathcal{A}^\bullet]^s$. The condition $(d_C + d_I)(\sum_J a'_J\ \varphi'_{j_1} \otimes \cdots \otimes \varphi'_{j_r}) = 0$ implies that T^s is a cocycle with respect to the differential of $(\bigotimes^s \mathcal{A}^\bullet,\ d)$. The Künneth formula ([Spa66], p. 228 and p. 247, Theorem 11) yields the short exact sequence (i. e. the isomorphism)

$$0 \longrightarrow \left[\bigotimes^s H^\bullet(\mathcal{A}^\bullet)\right]^s \longrightarrow H^s\left(\bigotimes^s \mathcal{A}^\bullet\right) \longrightarrow 0.$$

This is the reason why we can write T^s as $\sum_{|J|=s} a_J\ \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}$, where for each J with $|J| = s$ either all forms $\varphi_{j_1}, \ldots, \varphi_{j_s}$ are closed or at least one of them is exact. □

Suppose that $H^2(\mathcal{A}^\bullet) = 0$. Given closed elements $\varphi_1, \dots, \varphi_s \in \mathcal{A}^1$, the next proposition shows us, how we can find a Chen-closed element in $\bigoplus_{r=0}^{\infty} \bigotimes^r \mathcal{A}^1$, whose part in $\bigotimes^s \mathcal{A}^1$ is $\varphi_1 \otimes \cdots \otimes \varphi_s$.

Proposition 1.11 *Suppose $H^2(\mathcal{A}^\bullet) = 0$ and assume $\varphi_1, \dots, \varphi_s$ are closed elements in $\mathcal{A}^1$. Then there exist elements $\varphi_{i,\dots,j} \in \mathcal{A}^1$ (with $1 \le i < j \le s$) such that, putting $\varphi_{i,\dots,j} = \varphi_i$ if $i = j$, for all $1 \le i \le j \le s$:*

$$d\varphi_{i,i+1,\dots,j} + \sum_{k=i}^{j-1} \varphi_{i,\dots,k} \wedge \varphi_{k+1,\dots,j} = 0.$$

The following element in $\bigoplus_{r=1}^{s} \bigotimes^r \mathcal{A}^1$ is Chen-closed:

$$\sum_{\nu=1}^{s} \sum_{0<\alpha_1<\dots<\alpha_{\nu-1}<s} \varphi_{1,\dots,\alpha_1} \otimes \varphi_{\alpha_1+1,\dots,\alpha_2} \otimes \cdots \otimes \varphi_{\alpha_{\nu-1}+1,\dots,s}.$$

The proof of the existence of $\varphi_{i,i+1,\dots,j} \in \mathcal{A}^1$ goes by induction on $j - i$. The computations are just juggling with indices □

1.3 Homotopy Functionals

Denote by $E(M)$ or $E_{\mathbb{K}}(M)$ the differential graded algebra (dga) of $\mathbb{K}$-valued C^∞ differential forms on M. Note that any element $I = \sum_J a_J\ \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=0}^{s} \bigotimes^r E^1(M)$ defines an iterated integral $\sum_J a_J\ \int \varphi_{j_1} \cdots \varphi_{j_r}$, which we denote by $\int I$.

Lemma 1.12 *Let $\mathbb{T} = \sum_{|J|=s} a_J\ [\varphi_{j_1}] \otimes \cdots \otimes [\varphi_{j_s}] \in \bigotimes^s E^1(M)/dE^0(M)$ be such that holds:*

$$\sum_{|J|=s} a_J \int_{\alpha_1} \varphi_{j_1} \cdot \dots \cdot \int_{\alpha_s} \varphi_{j_s} = 0$$

for any s-tuple of closed paths $\alpha_1, \dots, \alpha_s$ in $P(M,p)$. Then: $\mathbb{T} = 0$.

Proof: Induction on s. In the case $s = 1$ the function $f(x) := \sum_{|J|=s} a_J\ \int_p^x \varphi_{j_1}$ is well defined and $df = \sum_{|J|=s} a_J\ \varphi_{j_1}$.

Now let $s > 1$: Choose a basis $[\omega_1], \dots, [\omega_N]$ for the finite dimensional subspace of $E^1(M)/dE^0(M)$, which is spanned by all elements $[\varphi_{j_\nu}]$ that appear in the expression for $\mathbb{T}$. Write

$$\mathbb{T} = \sum_{|J|=s} A_J\ [\omega_{j_1}] \otimes \cdots \otimes [\omega_{j_s}] = \sum_{\nu=1}^{N} [\omega_\nu] \otimes \left(\sum_{\substack{|J|=s \\ j_1=\nu}} A_J\ [\omega_{j_2}] \otimes \cdots \otimes [\omega_{j_s}] \right).$$

For any $(s-1)$-tuple $\alpha_2, \dots, \alpha_s$ of closed paths in $P(M,p)$ we have

$$= \sum_{\nu=1}^{N} [\omega_\nu] \cdot \left(\sum_{\substack{|J|=s \\ j_1=\nu}} A_J \int_{\alpha_2} \omega_{j_2} \cdot \dots \cdot \int_{\alpha_s} \omega_{j_s} \right) = 0 \in E^1(M)/dE^0(M),$$

since integration of this element in $E^1(M)/dE^0(M)$ over a closed path always yields zero. The $\{[\omega_\nu]\}_\nu$ form a basis and therefore we see that for any ν, the term

$$\mathbb{T}_\nu = \left(\sum_{\substack{|J|=s \\ j_1=\nu}} A_J \, [\omega_{j_2}] \otimes \dots \otimes [\omega_{j_s}] \right)$$

satisfies the assumptions of the induction hypothesis and is hence equal to 0. That proves $\mathbb{T} = 0$. □

The following theorem is due to K. T. Chen [Che77b]. We give a proof by using the universal covering space of M without using higher iterated integrals (i. e. with forms of degree higher than 1).

Theorem 1.13 (K. T. Chen) *Let I be an element of $\bigoplus_{r=0}^{s} \bigotimes^r E^1(M)$. If I is Chen-closed then $\int I$ is a homotopy functional.*

Proof: Let $\bar{E}^\bullet(M)$ be a sub-dga of $E^\bullet(M)$ like the dga $\bar{\mathcal{A}}^\bullet \subset \mathcal{A}^\bullet$ in (1.5) on page 19.

The proof goes by induction on s. For $\mathbf{s} = \mathbf{1}$ the assertion simply means that integration over a 1-form is a homotopy functional if this form is closed.
$\mathbf{s} > \mathbf{1}$: Let I be an element of $\bigoplus_{r=0}^{s} \bigotimes^r E^1(\tilde{M})$ and $\int I$ the corresponding iterated integral of length $\leq s$. Then let $I = \bar{I} + R$ be a decomposition of I according to Lemma 1.9. Because of $d_{\text{Chen}} I = 0 \Leftrightarrow (d_I + d_C)(\bar{I}) = 0$ on the one hand and $\int I = \int \bar{I}$ on the other hand, we may assume without loss of generality: $I = \bar{I}$ and $(d_I + d_C)(I) = 0$.

Now denote by $\pi : \tilde{M} \to M$ the universal covering and let $\tilde{\varphi}_{j_\nu}$ be the liftings $\pi^* \varphi_{j_\nu}$ as well as $\tilde{I} = \sum_{|J|=r \leq s} a_J \; \tilde{\varphi}_{j_1} \otimes \dots \otimes \tilde{\varphi}_{j_r}$ the lifting of I in $\bigoplus_{r=0}^{s} \bigotimes^r E^1(\tilde{M})$.

The iterated integral $\int I$ is a homotopy functional if and only if $\int \tilde{I}$ is one. And $(d_C + d_I)(I) = 0$ if and only if $(d_C + d_I)(\tilde{I}) = 0$. So, $(d_C + d_I)(\tilde{I}) = 0$ implies by Lemma 1.10 that we may assume without loss of generality that for any J with $|J| = s$ one of the forms $\tilde{\varphi}_{j_\nu}$ is closed. Since on $\tilde{M}$ all closed forms are exact, we know that for any such J at least one $\tilde{\varphi}_{j_i}$ is exact. By Proposition 1.3 for any pair of points p, $q \in M$ there is an element $\tilde{I}_{pq} \in \bigoplus_{r=0}^{s-1} \bigotimes^r E^1(\tilde{M})$ such that the restrictions of $\int \tilde{I}_{pq}$ and $\int \tilde{I}$ to $P(\tilde{M}; p, q)$ are equal. Using the relations (1.2), (1.3) and (1.4) one easily computes

$$d_{\text{Chen}}(\tilde{I}) = d_{\text{Chen}}(\tilde{I}_{pq}).$$

By induction hypothesis we conclude that $\int I$ is a homotopy functional. This completes the proof of the theorem. □

Also the following observation will turn out to be very useful later, especially likewise for $M = U$, an open simply connected neighbourhood of a point or for $M = \tilde{M}$, the universal covering.

Proposition 1.14 *Let M be a manifold with $H^1(M) = 0$. Let I be a Chen-closed element of $\bigoplus_{r=1}^{s} \bigotimes^r E^1(M)$.*
Then for any pair of points p, $q \in M$ there is a function $f_{pq} \in E^0(M)$ such that for all smooth paths $\gamma : [0;1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$ holds:

$$\int_\gamma I = \int_\gamma df_{pq} = f_{pq}(q) - f_{pq}(p).$$

Proof: By induction on s. $s = 1$: trivial.
$s \geq 1$: By Lemma 1.10 and Proposition 1.3 we know that we may express $\int_\gamma I$ as an iterated integral of length $s - 1$. The definition of all the forms in this shorter iterated integral just depends on the points p and q. Then apply the induction hypothesis. □

We will finish this section by describing the basic relation between iterated integrals and the fundamental group: Chen's theorem.

Let $\mathbb{Z}\pi_1(M,p)$ be the group ring of $\pi_1(M,p)$ with augmentation ideal $J_p(M)$ (if the context is clear we will only write J_p). Note that J_p is generated by elements of the form $(\alpha - 1)$, where $\alpha \in \pi_1(M,p)$. Notice moreover that we have the isomorphism

$$H_1(M,\mathbb{Z}) = \frac{\pi_1(M,p)}{[\pi_1(M,p);\pi_1(M,p)]} \xrightarrow{\cong} J_p/J_p^2,$$

which sends $[\alpha]$ to $(\alpha - 1)$.

Let $G_{\mathbf{K}}$ be a sub-dga of $E_{\mathbf{K}}(M)$. The choice of a base point $p \in M$ defines an augmentation map that we also denote by p:

$$\begin{array}{rccc} p: & G_{\mathbf{K}} & \to & \mathbb{K} \\ & f & \mapsto & f(p). \end{array}$$

By Proposition 1.3 and by Proposition 1.5 we have for all s well-defined integration maps:

$$H^0\bar{B}_s(G_{\mathbf{K}},p) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\left(J_p/J_p^{s+1};\mathbb{K}\right). \tag{1.6}$$

The following theorem is also due to Chen [Che77b] and is referred to as *Chen's π_1-De-Rham theorem* or *Chen's theorem* in the literature. It describes how iterated integrals reflect the structure of the fundamental group.

Theorem 1.15 (K. T. Chen) *Suppose that $G_{\mathbf{K}}$ is a sub-dga of $E_{\mathbf{K}}(M)$ such that the inclusion is a quasi-isomorphism. Then the integration map*

$$H^0\bar{B}_s(G_{\mathbf{K}},p) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\left(J_p/J_p^{s+1};\mathbb{K}\right)$$

is an isomorphism.

Remark 1.16 Observe that the augmentation map $\varepsilon : \mathbb{Z}\pi_1(M,p) \longrightarrow \mathbb{Z}$ has a natural section $\sigma : n \longmapsto n \cdot \mathbf{1}$, which makes the short exact sequence

$$0 \longrightarrow J_p/J_p^{s+1} \longrightarrow \mathbb{Z}\pi_1(M,p)/J_p^{s+1} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

split. This splitting gives

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}\pi_1(M,p)/J_p^{s+1};\mathbb{K}\right) \cong \mathrm{Hom}_{\mathbb{Z}}\left(J_p/J_p^{s+1};\mathbb{K}\right) \oplus \mathbb{K}$$

and this decomposition corresponds via the integration map with

$$H^0B_s(G_{\mathbb{K}},p) \cong H^0\bar{B}_s(G_{\mathbb{K}},p) \oplus \mathbb{K}, \tag{1.7}$$

where $\mathbb{K}$ stands for the iterated integrals of length 0, i. e. the constants.

Therefore Chen's theorem can also be stated as: The integration map induces an isomorphism

$$\boxed{H^0B_s(G_{\mathbb{K}},p) \cong \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}\pi_1(M,p)/J_p^{s+1};\mathbb{K}\right).}$$

Remark 1.17 Taking Chen's Theorem 1.15 for granted one can prove in particular the converse of Theorem 1.13: if $\int I$ is a homotopy functional then I is Chen-closed.

About the proof of 1.15: It is a consequence of Lemma 1.12 and Proposition 1.5 that if for an element $I \in H^0\bar{B}_s(G_{\mathbb{K}},p)$ the integral $\int I : J_p^s \to \mathbb{K}$ is the zero map then $I \in H^0\bar{B}_{s-1}(G_{\mathbb{K}},p)$. Conversely, if $I \in H^0\bar{B}_{s-1}(G_{\mathbb{K}},p)$ then $\int I : J_p^s \to \mathbb{K}$ is the zero map. This shows that there is an injective map

$$\mathfrak{I} : \frac{H^0\bar{B}_s(G_{\mathbb{K}},p)}{H^0\bar{B}_{s-1}(G_{\mathbb{K}},p)} \longrightarrow \left(J_p^s/J_p^{s+1}\right)^*. \tag{1.8}$$

Consider for all s the short exact sequence

$$0 \longrightarrow J_p^s/J_p^{s+1} \longrightarrow J_p/J_p^{s+1} \longrightarrow J_p/J_p^s \longrightarrow 0.$$

Dualizing with $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,;\mathbb{K})$ yields the lower sequence of the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & H^0\bar{B}_{s-1}(G_{\mathbb{K}},p) \arrow[r] \arrow[d] & H^0\bar{B}_s(G_{\mathbb{K}},p) \arrow[r] \arrow[d] & \frac{H^0\bar{B}_s(G_{\mathbb{K}},p)}{H^0\bar{B}_{s-1}(G_{\mathbb{K}},p)} \arrow[r] \arrow[d, "\mathfrak{I}"] & 0 \\
0 \arrow[r] & (J_p/J_p^s)^*_{\mathbb{K}} \arrow[r] & (J_p/J_p^{s+1})^*_{\mathbb{K}} \arrow[r] & (J_p^s/J_p^{s+1})^*_{\mathbb{K}} \arrow[r] & 0.
\end{tikzcd}$$

By induction on s using the 5-lemma we obtain another equivalent formulation of Chen's theorem: For all s, the integration map induces an isomorphism:

$$\mathfrak{I} : \frac{H^0\bar{B}_s(G_{\mathbb{K}},p)}{H^0\bar{B}_{s-1}(G_{\mathbb{K}},p)} \longrightarrow \left(J_p^s/J_p^{s+1}\right)^*.$$

The multiplication of the group ring induces a surjective map:

$$\bigotimes^s J_p/J_p^2 \longrightarrow J_p^s/J_p^{s+1}.$$

By dualizing this map we find an embedding

$$\iota : (J_p^s/J_p^{s+1})^*_{\mathbf{K}} = \mathrm{Hom}_{\mathbf{Z}}(J_p^s/J_p^{s+1};\mathbb{K}) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}\left((J_p/J_p^2)^{\otimes s};\mathbb{K}\right) = H^1(X,\mathbb{K})^{\otimes s}.$$

We do not prove the surjectivity of $\mathfrak{I}$ in general. But, for example, if $H^2(M) = 0$, then we know by Proposition 1.11 that even $\iota \circ \mathfrak{I}$ is surjective and so is $\mathfrak{I}$.

In particular, this proves Chen's theorem for a punctured Riemann surface, i. e. the complement of one point in a compact Riemann surface. For compact Riemann surfaces, we will prove the surjectivity of $\mathfrak{I}$ explicitly in Proposition 2.2 and Proposition 2.8 (cf. Remark 2.9). □

Chapter 2

De Rham Homotopy Theory for Riemann Surfaces

In this chapter we recapitulate the De Rham Homotopy theory in the sense of Hain for the case of a compact Riemann surface and the case of the complement of a point within a compact Riemann surface. We construct the *MHS on the fundamental group* in these cases. This is all classical. As reference may serve: [Hai87c].

2.1 The Topology

In the sequel we let X be a compact Riemann surface of genus g. We fix $p \in X$ as basepoint for the fundamental group and $q \in X$ as the point, which will possibly be removed. Since several statements are analogous for X and $X \setminus \{q\}$ we write S for an element of the set $\{X,\ X \setminus \{q\}\}$.

The goal of this section is to understand Chen's theorem better in the two cases $(S,p) = (X,p)$ and $(S,p) = (X \setminus \{q\}, p)$. In particular, we want to get a description of $J_p(S)/J_p(S)^{s+1}$.

Since $J_p(S)$ is the augmentation ideal in $\mathbb{Z}\pi_1(S,p)$, the multiplication in $\mathbb{Z}\pi_1(S,p)$ defines a surjective map

$$\bigotimes^s J_p(S)/J_p(S)^2 \twoheadrightarrow J_p(S)^s/J_p(S)^{s+1},$$

whose dual gives us a natural inclusion ("$(\cdot)^* = \mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$")

$$\left(J_p(S)^s/J_p(S)^{s+1}\right)^* \hookrightarrow \bigotimes^s H^1(S,\mathbb{Z}).$$

Hence, the exact sequence

$$0 \to J_p^s/J_p^{s+1} \to J_p/J_p^{s+1} \to J_p/J_p^s \to 0$$

induces the exact sequence

$$0 \to \left(J_p/J_p^s\right)^* \to \left(J_p/J_p^{s+1}\right)^* \xrightarrow{P} \bigotimes^s H^1(S,\mathbb{Z}). \tag{2.1}$$

2.1.1 The Punctured Riemann Surface

When $S = X \setminus \{q\}$, let $J_{pq} := J_p(X \setminus \{q\})$. Here we have $H^2(X \setminus \{q\}) = 0$ and Proposition 1.11 tells us that we can find for any $\mathbb{T} = \sum_{|J|=s} a_J[\varphi_{j_1}]\otimes\cdots\otimes[\varphi_{j_s}] \in \bigotimes^s H^1(X\setminus\{q\},\mathbb{Z})$ a Chen-closed element $I \in \bigoplus_{r=1}^s \bigotimes^r E^1(X\setminus\{q\})$, whose part in $\bigotimes^s E^1(X \setminus \{q\})$ is $\sum_{|J|=s} a_J\varphi_{j_1} \otimes\cdots\otimes \varphi_{j_s}$. When we use Chen's theorem to identify $H^0\bar{B}(E^\bullet(X \setminus \{q\}),p)$ with $\left(J_{pq}/J_{pq}^{s+1}\right)^*$, then we find that P maps I to $\mathbb{T}$. This shows that P is surjective in this case. We proved:

Proposition 2.1 *There is a natural exact sequence*

$$0 \to \left(J_{pq}/J_{pq}^s\right)^* \to \left(J_{pq}/J_{pq}^{s+1}\right)^* \xrightarrow{P} \bigotimes^s H^1(X \setminus \{q\},\mathbb{Z}) \to 0.$$

2.1.2 The Compact Riemann Surface

Now consider $S = X$ and let $J_p := J_p(X)$. The wedge product on $E_{\mathbf{K}}(X)$ induces the cup product

$$\cup : H^1(X,\mathbb{Z}) \otimes H^1(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z}).$$

Denote its kernel by $K(X,\mathbb{Z})$ and define the following submodule of $\bigotimes^s H^1(X,\mathbb{Z})$ (abbreviate $H^1(X,\mathbb{Z}) = H^1(X)$):

$$K^s(X,\mathbb{Z}) := \bigcap_{i=0}^{s-2} \underbrace{H^1(X)\otimes\cdots\otimes H^1(X)}_{i} \otimes K(X,\mathbb{Z}) \otimes \underbrace{H^1(X)\otimes\cdots\otimes H^1(X)}_{(s-2)-i}.$$

Or alternatively, for $s \geq 2$ the cup-product induces maps

$$\cup_i : H^1(X)^{\otimes s} \to H^1(X)^{\otimes i} \otimes H^2(X) \otimes H^1(X)^{\otimes(s-2-i)}$$

for $i = 1,\ldots,s-2$. Then $K^s(X,\mathbb{Z}) := \bigcap_{i=0}^{s-2} \ker \cup_i$ and $K^2(X,\mathbb{Z}) = K(X,\mathbb{Z})$.
In this subsection, we want to prove the following proposition.

Proposition 2.2 *There is a natural exact sequence*

$$0 \to \left(J_p/J_p^s\right)^* \to \left(J_p/J_p^{s+1}\right)^* \xrightarrow{P} K^s(X,\mathbb{Z}) \to 0.$$

Let us introduce some notation, which we will also use later on. Denote by $\underline{\gamma}_1,\ldots\underline{\gamma}_{2g}$ a system of closed piecewise smooth paths in $X \setminus \{q\}$ with basepoint p, disjoint in $X \setminus \{p,q\}$, which has the following properties:

- Let $\pi : \tilde{X} \longrightarrow X$ denote the universal covering space of X. Then a lifting of the path

$$\underline{\gamma}_1 \underline{\gamma}_{g+1} \underline{\gamma}_1^{-1} \underline{\gamma}_{g+1}^{-1} \cdots \underline{\gamma}_g \underline{\gamma}_{2g} \underline{\gamma}_g^{-1} \underline{\gamma}_{2g}^{-1}$$

to $\tilde{X}$ parametrizes the boundary of a disk in $\tilde{X}$.

- Denote by γ_i the homotopy classes of the paths $\underline{\gamma}_i$, $i = 1, \dots, 2g$. The fundamental group $\pi_1(X,p)$ is the quotient of the free group $F\langle\gamma_1, \dots, \gamma_{2g}\rangle$, which is generated by the γ_i, and the only relation (let $[\cdot,\ \cdot]$ denote the commutator)

$$[\gamma_1,\ \gamma_{g+1}] \cdots [\gamma_g,\ \gamma_{2g}] = 1. \tag{2.2}$$

By $c_i := (\gamma_i - 1)$ we denote the element in J_p, which corresponds to $\gamma_i \in \pi_1(X,p)$. We make the convention that we write greek letters for elements in $\pi_1(X,p)$ and latin letters for the corresponding element in J_p, e. g. $\alpha \in \pi_1(X,p)$ and $a = (\alpha - 1) \in J_p$.

With $a = (\alpha - 1)$, $b = (\beta - 1)$, $c = (\gamma - 1)$ and $d = (\delta - 1)$ we find in general $(\alpha\beta\gamma - 1) = (a+1)(b+1)(c+1) - 1$ resp.

$$(\alpha\beta\gamma - 1) \equiv ab + ac + bc + a + b + c \mod J_p^3$$

$$(\beta\alpha\gamma - 1) \equiv ba + ac + bc + a + b + c \mod J_p^3$$

from which we obtain

$$(\alpha\beta\gamma - 1) - (\beta\alpha\gamma - 1) \equiv ab - ba \mod J_p^3.$$

Hence for $\gamma = \alpha^{-1}\beta^{-1}\delta$ we find:

$$(\alpha\beta\alpha^{-1}\beta^{-1}\delta - 1) - (\delta - 1) \equiv ab - ba \mod J_p^3.$$

Applying this rule to the relation (2.2) yields:

$$\sum_{i=1}^{g} (c_i c_{g+i} - c_{g+i} c_i) \equiv 0 \mod J_p^3. \tag{2.3}$$

Deduce from $(\alpha\beta - 1) = ab + a + b$ and in particular $(\gamma_i - 1)(\gamma_i^{-1} - 1) + (\gamma_i - 1) + (\gamma_i^{-1} - 1) = 0$ that J_p^s / J_p^{s+1} is generated as $\mathbb{Z}$-module by all s-fold products of $c_1, \dots, c_{2g}$. Relation (2.3) says that there is a formal power series $P(x_1, \dots, x_{2g}) = \sum_{|J| \geq 3} a_J\, x_{j_1} \cdots x_{j_r} \in \mathbb{Z}[[x_1, \dots, x_{2g}]]$ in such a way that

$$[\gamma_1,\ \gamma_{g+1}] \cdots [\gamma_g,\ \gamma_{2g}] - 1 = \sum_{i=1}^{g} (c_i c_{g+i} - c_{g+i} c_i) + P(c_1, \dots, c_{2g}) = 0.$$

There are relations between these s-fold products of c_j's: the relations coming from (2.2):

$$v\left(\sum_{i=1}^{g}(c_i c_{g+i} - c_{g+i}c_i)\right) w \equiv 0 \mod J_p^{s+1}$$

where v, w are products of the c_j's with $v \in J_p^k$ and $w \in J_p^{s-2-k}$, $k = 0, \dots, s-2$ and $J_p^0 := \mathbb{Z}\,1$. These are the only relations as one can derive from

$$\mathbb{Z}\pi_1(X,p) \cong \frac{\mathbb{Z}F\langle\gamma_1, \dots \gamma_{2g}\rangle}{([\gamma_1,\ \gamma_{g+1}]\cdots[\gamma_g,\ \gamma_{2g}] - 1)}, \tag{2.4}$$

where the denominator is the ideal generated by $[\gamma_1,\ \gamma_{g+1}]\cdots[\gamma_g,\ \gamma_{2g}]-1$. (This isomorphism exists since the object on the right in (2.4) satisfies the universal property of the group ring (cf. [Lan93]Ch. V, §1, p. 176).)

The observations above can be summarized in the following lemma.

Lemma 2.3 *The kernel of the multiplication map*

$$\bigotimes^s J_p/J_p^2 \quad \longrightarrow \quad J_p^s/J_p^{s+1}.$$

is generated by elements of the form: $x \otimes \sum_{i=1}^{g}(c_i \otimes c_{g+i} - c_{g+i} \otimes c_i) \otimes y$, *where* x *is in* $(J_p/J_p^2)^{\otimes k}$ *and* y *in* $(J_p/J_p^2)^{\otimes(s-2)-k}$ *for* $k = 0, \dots, s-2$.

Let $[\mathfrak{X}] \in H^2(X,\mathbb{Z})$ be a generator such that $\int_X \mathfrak{X} = 1$. Then the first reciprocity law for Riemann surfaces ([GH78], p. 230) yields the following equation for two elements $[\varphi]$, $[\psi] \in H^1(X)$:

$$\begin{aligned}
[\varphi] \cup [\psi] &= (\int_X \varphi \wedge \psi)[\mathfrak{X}] \\
&= \left\{\sum_{\nu=1}^{g}\left(\int_{c_\nu}\varphi\int_{c_{g+\nu}}\psi - \int_{c_{g+\nu}}\varphi\int_{c_\nu}\psi\right)\right\}[\mathfrak{X}].
\end{aligned}$$

In other words, the map $\rho : H_2(X) \to H_1(X) \otimes H_1(X)$, which maps $[X]$ to $\sum_{\nu=1}^{g}(c_\nu \otimes c_{g+\nu} - c_{g+\nu} \otimes c_\nu)$, is the dual to the cup-product $\cup : H^1(X,\mathbb{Z}) \otimes H^1(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$.

Proof of Proposition 2.2: In Lemma 2.3 we saw that the sequence over $\mathbb{Z}$,

$$0 \to \bigoplus_{i=1}^{s-2} H_1(X) \otimes \cdots \otimes \underset{\stackrel{\uparrow}{i}}{H_2(X)} \otimes \cdots H_1(X) \to \bigotimes^s H_1(X) \to (J_p^s/J_p^{s+1}) \to 0,$$

where the first map is induced by ρ, is exact. Its dual is the short exact sequence

$$0 \to (J_p^s/J_p^{s+1})^* \to \bigotimes^s H^1(X) \xrightarrow{C} \bigoplus_{i=1}^{s-2} H^1(X) \otimes \cdots \otimes \underset{\stackrel{\uparrow}{i}}{H^2(X)} \otimes \cdots H^1(X) \to 0,$$

where the last map C is induced by the cup-product.

Hence we have: $(J_p^s/J_p^{s+1})^* = \ker C = K^s(X,\mathbb{Z})$. $\square$

2.2 The Mixed Hodge Structure on the Fundamental Group

In this section we describe the construction of the "mixed Hodge structure (MHS) on the fundamental group" in the two cases we are interested in: the case of a compact Riemann surface and the case of a Riemann surface, which is the complement of one point in a compact Riemann surface. This time, we start with the compact case.

2.2.1 Compact Riemann Surface

By $\mathbb{K}$ we denote the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Let $E^\bullet_{\mathbb{K}}(X)$ or $E^\bullet_{\mathbb{K}}$ be the algebra of $\mathbb{K}$-valued C^∞ forms on X. The *weight filtration* $W_\bullet$ on $E^\bullet_{\mathbb{R}}(X)$ is defined to be the increasing filtration for which holds: $Gr_0^W E^\bullet_{\mathbb{R}}(X) = E^\bullet_{\mathbb{R}}(X)$.

Moreover, there is a decreasing filtration $F^\bullet$ on $E^\bullet_{\mathbb{C}}(X) = E^\bullet_{\mathbb{R}}(X) \otimes_{\mathbb{R}} \mathbb{C}$, *the Hodge filtration*: on $E^\bullet_{\mathbb{C}}(X)$, the subspace $F^pE(X)$ contains by definition all forms, with at least p dz's, when the forms are represented by local coordinates (U, z). The only non trivial part of this filtration is hence $F^1E^1(X)$. The filtration $F^\bullet$ has the following well-known properties.

$$E^\bullet(X) := ((E^\bullet_{\mathbb{R}}, W), (E^\bullet_{\mathbb{C}}, W, F))$$

is an $\mathbb{R}$-Hodge complex[1] of weight 0. This implies in particular that $((H^n(X,\mathbb{R}), W_\bullet), (H^n(X,\mathbb{C}), W_\bullet, F^\bullet))$ is a Hodge structure (HS) of weight n.

The wedge product on $E^\bullet(X)$ induces a morphism of Hodge complexes:

$$\wedge : E^\bullet(X) \otimes E^\bullet(X) \longrightarrow E^\bullet(X).$$

Therefore, the cup product $\cup : H^1(X) \otimes H^1(X) \longrightarrow H^2(X)$ defines a morphism of Hodge structures. Thus, its kernel

$$K(X) := (K(X,\mathbb{Z}),\ (K(X,\mathbb{R}), W),\ (K(X,\mathbb{C}), W, F))$$

is a $\mathbb{Z}$-Hodge structure of weight 2. More generally,

$$K^s(X) := (K^s(X,\mathbb{Z}),\ (K^s(X,\mathbb{R}), W),\ (K^s(X,\mathbb{C}), W, F))$$

is a $\mathbb{Z}$-Hodge structure of weight s.

Our goal now is to define for all $s \geq 1$ a mixed Hodge structure (MHS) on $(J_p/J_p^{s+1})^*$ in such a way that the exact sequences

$$0 \to (J_p/J_p^s)^* \to (J_p/J_p^{s+1})^* \to K^s(X) \to 0$$

become exact sequences of mixed Hodge structures. The way to do this, is to define a MHS on $H^0\bar{B}_s(E^\bullet(X), p)$ and to use Chen's π_1-De-Rham theorem to define a MHS on $(J_p/J_p^{s+1})^*$.

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[1]For the notion of a Hodge complex (HC) and a mixed Hodge complex (MHC) we refer to [Del71] or [EZ91].

The filtrations $W_\bullet$ and $F^\bullet$ on $E^\bullet(X)$ induce filtrations $W_\bullet$ and $F^\bullet$ on $\bigoplus_{r=1}^s \bigotimes^r E^\bullet(X)$ and also on $\bar{\mathcal{K}}^s(E^\bullet(X),p)$ (see Definition 1.7 on page 19) in a natural way by

$$W_l\left(\bigoplus_{r=1}^{s}\bigotimes^{r} E^\bullet_{\mathbb{R}}(X)\right) = \bigoplus_{r=1}^{s} \sum_{l_1+\cdots+l_r+r\le l} W_{l_1}E^\bullet_{\mathbb{R}}(X)\otimes\cdots\otimes W_{l_r}E^\bullet_{\mathbb{R}}(X)$$

and

$$F^p\left(\bigoplus_{r=1}^{s}\bigotimes^{r} E^\bullet_{\mathbb{C}}(X)\right) = \bigoplus_{r=1}^{s} \sum_{p_1+\cdots+p_r\ge p} F^{p_1}E^\bullet_{\mathbb{C}}(X)\otimes\cdots\otimes F^{p_r}E^\bullet_{\mathbb{C}}(X).$$

Then define

$$W_l\left(H^0\bar{B}_s(E^\bullet_{\mathbb{R}}(X),p)\right) := \operatorname{im}\left\{W_l\bar{\mathcal{K}}^s(E^\bullet_{\mathbb{R}}(X),p) \to H^0\bar{B}_s(E^\bullet_{\mathbb{R}}(X),p)\right\}$$

and

$$F^p\left(H^0\bar{B}_s(E^\bullet_{\mathbb{C}}(X),p)\right) := \operatorname{im}\left\{F^p\bar{\mathcal{K}}^s(E^\bullet_{\mathbb{R}}(X),p) \to H^0\bar{B}_s(E^\bullet_{\mathbb{R}}(X),p)\right\}$$

Note that for $s \ge l$ we have $W_l\left(H^0\bar{B}_s(E^\bullet_{\mathbb{R}}(X),p)\right) = H^0\bar{B}_l(E^\bullet_{\mathbb{R}}(X),p)$, i. e. the weight filtration is just *the filtration by the length of the iterated integrals.*

We want to prove the following theorem.

Theorem 2.4 *For all $s \ge 1$, the collection of data*

$$H^0\bar{B}_s(X,p) := \left(\left(H^0\bar{B}_s(E^\bullet_{\mathbb{R}}(X),p),\, W_\bullet\right),\, \left(H^0\bar{B}_s(E^\bullet_{\mathbb{C}}(X),p),\, W_\bullet,\, F^\bullet\right)\right)$$

is an $\mathbb{R}$-mixed Hodge structure.

To this end, the following lemma turns out to be very useful (cf. [Pul88], section 3 and [PY96], section 2, Lemma 2.4). Define for $\Omega = \sum_\iota \varphi_\iota \otimes \psi_\iota \in E^1_{\mathbb{C}}(X)\otimes E^1_{\mathbb{C}}(X)$ the notation: $\wedge\Omega := \sum_\iota \varphi_\iota \wedge \psi_\iota$.

Lemma 2.5 *There is a unique linear map*

$$\mu : \left\{\Omega \in E^1_{\mathbb{C}}(X)\otimes E^1_{\mathbb{C}}(X) \,\middle|\, \wedge\,\Omega \textit{ is exact}\right\} \longrightarrow F^1E^1_{\mathbb{C}}(X)$$

such that for all $\varphi\otimes\psi$ with $\varphi\wedge\psi$ exact holds:

$$\varphi\wedge\psi + d\mu(\varphi\otimes\psi) = 0 \textit{ and } \mu(\varphi\otimes\psi) - \overline{\mu(\bar\varphi\otimes\bar\psi)} \textit{ is exact.}$$

Remark 2.6 If one uses in the case $S = X\setminus\{q\}$ the C^∞-logarithmic Dolbeault complex of differential forms, which was introduced and studied by Burgos [Bur94], then one can also prove a comparable lemma for the case $S = X\setminus\{q\}$.

However, in 2.2.2 we will – like it is done in [Hai87c] – use $E^\bullet(X\log q)$, the complex of C^∞-forms on X with logarithmic singularities at q in order to construct a MHS on $(J_{pq}/J_{pq}^{s+1})^*$.

Proof: The following assertion provides the proof:
Assertion: *For each $\Omega = \sum_\iota \varphi_\iota \otimes \psi_\iota \in E^1_{\mathbb{C}}(X) \otimes E^1_{\mathbb{C}}(X)$ such that $\wedge\Omega = \sum_\iota \varphi_\iota \wedge \psi_\iota$ is exact, there exists a unique pair $\mu(\Omega)$ and $\tilde{\mu}(\Omega)$ in $F^1E^1_{\mathbb{C}}(X)$ with the properties:*

$$\wedge\Omega + d\mu(\Omega) = 0 \qquad \wedge\bar{\Omega} + d\tilde{\mu}(\Omega) = 0$$

and $\mu(\Omega) - \overline{\tilde{\mu}(\Omega)}$ is exact.
Proof of the assertion: Since the differential of $E^\bullet_{\mathbb{C}}(X)$ is strict (cf. [GH78]) with respect to the Hodge filtration $F^\bullet$, there exist forms $v, \tilde{v} \in F^1E^1_{\mathbb{C}}(X)$ such that $\wedge\Omega + dv = 0$ and $\overline{\wedge\Omega} + d\tilde{v} = 0$.

Now $v - \bar{\tilde{v}}$ is a closed form, i. e. it represents an element in $H^1(E^\bullet_{\mathbb{C}})$. Recall that the Hodge filtration F on $E^\bullet_{\mathbb{C}}$ induces a pure Hodge structure of weight 1 on $H^1(E^\bullet_{\mathbb{C}})$. Therefore there are closed w, $\tilde{w} \in E^{1,0}_{\mathbb{C}} = F^1E^1_{\mathbb{C}}(X) \cap \bar{F}^0E^1_{\mathbb{C}}(X)$ and $f \in E^0_{\mathbb{C}}$ with: $v - \bar{\tilde{v}} = w - \bar{\tilde{w}} + df$. Define $\mu(\Omega) := v - w$ and $\tilde{\mu}(\Omega) := \tilde{v} - \tilde{w}$. Then $\mu(\Omega) - \overline{\tilde{\mu}(\Omega)} = (v - \bar{\tilde{v}}) - (w - \bar{\tilde{w}})$ is exact.
Uniqueness: Suppose there are $\mu'(\Omega)$ and $\tilde{\mu}'(\Omega)$ with the same properties. On cohomology we have $F^0H^1(E^\bullet_{\mathbb{C}}) = H^1(E^\bullet_{\mathbb{C}})$. Then $\mu(\Omega) - \mu'(\Omega)$ and $\tilde{\mu}(\Omega) - \tilde{\mu}'(\Omega)$ are closed and represent elements in $F^1H^1(E^\bullet_{\mathbb{C}}) = F^1H^1(E^\bullet_{\mathbb{C}}) \cap \bar{F}^0H^1(E^\bullet_{\mathbb{C}}) = H^{1,0}(E^\bullet_{\mathbb{C}})$ and $(\mu(\Omega) - \mu'(\Omega)) - (\overline{\tilde{\mu}(\Omega)} - \overline{\tilde{\mu}'(\Omega)})$ is exact. Therefore we obtain the decomposition

$$0 = [\mu(\Omega) - \mu'(\Omega)] + [\overline{\tilde{\mu}'(\Omega)} - \overline{\tilde{\mu}(\Omega)}] \in H^{1,0}(E^\bullet_{\mathbb{C}}) \oplus H^{0,1}(E^\bullet_{\mathbb{C}}).$$

Hence there exist f and $\tilde{f} \in E^0_{\mathbb{C}}$ with the property: $\mu(\Omega) - \mu'(\Omega) = df$ resp. $\tilde{\mu}(\Omega) - \tilde{\mu}'(\Omega) = d\tilde{f}$. Since d is strictly compatible with the Hodge filtration we observe $df, d\tilde{f} \in F^1E^1_{\mathbb{C}} \cap dE^0_{\mathbb{C}} = dF^1E^0_{\mathbb{C}} = 0$. This accomplishes the proof of the assertion.

The assertion implies $\mu(\bar{\Omega}) = \tilde{\mu}(\Omega)$, because $(\mu(\Omega), \mu(\bar{\Omega}))$ and $(\mu(\Omega), \tilde{\mu}(\Omega))$ both satisfy the conditions in the assertion. So, we constructed one map:

$$\mu : \{\Omega \in E^1_{\mathbb{C}}(X) \otimes E^1_{\mathbb{C}}(X) \,|\, \wedge\Omega \text{ exact }\} \longrightarrow F^1E^1_{\mathbb{C}}(X).$$

From the uniqueness of the assertion we also deduce that μ is linear, for with $(\mu(\Omega_1 + \Omega_2), \mu(\overline{\Omega_1 + \Omega_2}))$ also $(\mu(\Omega_1) + \mu(\Omega_2), \mu(\overline{\Omega_1}) + \mu(\overline{\Omega_2}))$ is a solution of the problem formulated in the assertion. This completes the proof of the lemma. □

Corollary 2.7 *For any form $\omega \in E^1_{\mathbb{C}}(X)$ with $\partial\omega = \bar{\partial}\omega = 0$ and any $\Omega \in E^1_{\mathbb{C}}(X) \otimes E^1_{\mathbb{C}}(X)$ such that $\wedge\Omega$ is exact,*

$$\omega \wedge \mu(\Omega) \text{ is exact.}$$

Proof: Decompose $\omega \in E^1_{\mathbb{C}}(X) = F^1E^1_{\mathbb{C}}(X) \oplus \bar{F}^1E^1_{\mathbb{C}}(X)$ in $\omega = \omega^{1,0} + \omega^{0,1}$. From the lemma we know that there is an $f \in E^0_{\mathbb{C}}$ such that $\mu(\bar{\Omega}) - \overline{\mu(\Omega)} = df$. Since $\mu(\Omega)$ and $\mu(\bar{\Omega})$ are in $F^1E^1_{\mathbb{C}}(X)$, we obtain:

$$\begin{aligned} \omega \wedge \mu(\Omega) &= \omega^{0,1} \wedge \mu(\Omega) \\ &= \overline{\overline{\omega^{0,1}} \wedge \mu(\bar{\Omega}) - \overline{\omega^{0,1}} \wedge df} \\ &= 0 + d\left(\bar{f}\omega^{0,1}\right) \square \end{aligned}$$

Consider the map $R : H^0\bar{B}_s(E^\bullet(X),p) \to K^s(X)$, which is induced by the projection $\bigoplus_{r=1}^s \bigotimes^r E^1(X) \to \bigotimes^s E^1(X)$. It is easy to see that the integration map makes the diagram

$$\begin{array}{ccc} H^0\bar{B}_s(E^\bullet(X),p) & \xrightarrow{R} & K^s(X) \\ \downarrow & & \| \\ (J_p/J_p^{s+1})^* & \xrightarrow{P} & K^s(X) \end{array}$$

commutative.

Proposition 2.8 *The map*

$$R : \left(H^0\bar{B}_s(E^\bullet(X),p),\, W_\bullet,\, F^\bullet\right) \to \left(K^s(X),\, W_\bullet,\, F^\bullet\right)$$

is surjective and strict with respect to the filtrations $W_\bullet$ and $F^\bullet$.

Proof: Define

$$\tilde{K}^r := \left\{ \sum_J a_J\, \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigotimes^r E^1_{\mathbb{C}}(X) \;\middle|\; \begin{array}{c} \forall J : \varphi_{j_k} \wedge \varphi_{j_{k+1}} \text{ is exact and} \\ (\varphi_{j_\nu} \in \operatorname{im}\mu \text{ or } \partial\varphi_{j_\nu} = \bar{\partial}\varphi_{j_\nu} = 0) \end{array} \right\}$$

and define moreover $\mu : \bigoplus_{r\geq 2} \tilde{K}^r \longrightarrow \bigoplus_{r\geq 1} \tilde{K}^r$ by fixing the values on elements of the form $\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}$ and extending linearly:

$$\mu(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}) := \sum_{k=1}^{r-1} \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_{k-1}} \otimes \mu(\varphi_{j_k} \otimes \varphi_{j_{k+1}}) \otimes \varphi_{j_{k+2}} \otimes \cdots \otimes \varphi_{j_r}.$$

Corollary 2.7 assures that this expression is an element of $\tilde{K}^{r-1}$ (notice that $F^1E^1_{\mathbb{C}}(X) \wedge F^1E^1_{\mathbb{C}}(X) = 0$).

We claim that for harmonic forms $\omega_1, \ldots, \omega_s \in E^1_{\mathbb{C}}(X)$, that is $\partial\omega_\nu = \bar{\partial}\omega_\nu = 0$, the expression

$$I := \omega_1 \otimes \cdots \otimes \omega_s + \mu(\omega_1 \otimes \cdots \otimes \omega_s) + \mu^2(\omega_1 \otimes \cdots \otimes \omega_s) + \cdots + \mu^{s-1}(\omega_1 \otimes \cdots \otimes \omega_s)$$

is Chen-closed and hence by Theorem 1.13, $\int I$ is a homotopy functional.

Observe that always holds:

$$d_C\left(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}\right) + d_I\left(\mu(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r})\right) = 0.$$

Therefore we find

$$(d_I + d_C)\left(\omega_1 \otimes \cdots \otimes \omega_s + \mu(\omega_1 \otimes \cdots \otimes \omega_s) + \cdots + \mu^{s-1}(\omega_1 \otimes \cdots \otimes \omega_s)\right)$$

$$= d_I\left(\omega_1 \otimes \cdots \otimes \omega_s\right) + d_C\left(\mu^{s-1}(\omega_1 \otimes \cdots \otimes \omega_s)\right) = 0.$$

This shows that R is surjective. Since μ respects the Hodge - and the weight filtration, the map R is strict with respect to $W_\bullet$ and $F^\bullet$. □

Remark 2.9 In particular, this proposition provides a proof of Chen's π_1-De-Rham theorem for compact Riemann surfaces, like mentioned before on p.25.

It is not difficult anymore to do all the remaining verifications in the proof of the following proposition by using Proposition 1.3.

Proposition 2.10 *For all s the sequence*

$$0 \to H^0\bar{B}_{s-1}(E^\bullet(X),p) \to H^0\bar{B}_s(E^\bullet(X),p) \xrightarrow{R} K^s(X) \to 0$$

is exact and strict with respect to $W_\bullet$ and $F^\bullet$. □

Now finally, we can prove Theorem 2.4.

Proof of 2.4: We prove the theorem by induction on s.

$s = 1$: Here we know: $H^0\bar{B}_1(E^\bullet(X),p) = H^1(X)$.

$s > 1$: Consider the exact sequence

$$0 \to H^0\bar{B}_{s-1}(E^\bullet(X),p) \to H^0\bar{B}_s(E^\bullet(X),p) \xrightarrow{R} K^s(X) \to 0.$$

By induction hypothesis, we have MHSs on the left and on the right hand side of this sequence. The maps are defined over $\mathbb{Z}$ and strict with respect to $W_\bullet$ and $F^\bullet$. It is follows (cf. [GS75], (1.16)) that $H^0\bar{B}_s(E^\bullet(X),p)$ is a real MHS. □

Remark 2.11 Together with Proposition 2.10 this theorem implies that the short exact sequence of 2.10 is an element of

$$\mathrm{Ext}_{\mathrm{MHS}}\left(K^s(X),\, H^0\bar{B}_{s-1}(E^\bullet(X),p)\right).$$

In particular for the case $s = 2$ we find an element in: $\mathrm{Ext}_{\mathrm{MHS}}\left(K^s(X),\, H^1(X)\right)$.

For $\mathbb{A} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ let

$$\left(J_p/J_p^{s+1}\right)^*_{\mathbb{A}} := \mathrm{Hom}_{\mathbb{Z}}\left(J_p/J_p^{s+1},\, \mathbb{A}\right).$$

By Chen's theorem, the filtrations $W_\bullet$ and $F^\bullet$ on $H^0\bar{B}_s(E^\bullet(X),p)$ induce filtrations $W_\bullet$ and $F^\bullet$ on $\left(J_p/J_p^{s+1}\right)^*_{\mathbb{A}}$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}$ respectively.

Therefore, we can summarize the results of this subsection into the following theorem.

Theorem 2.12 *The collection of data*

$$\left(J_p/J_p^{s+1}\right)^* := \left(\left(J_p/J_p^{s+1}\right)^*_{\mathbb{Z}};\, \left(\left(J_p/J_p^{s+1}\right)^*_{\mathbb{Q}}, W_\bullet\right);\, \left(\left(J_p/J_p^{s+1}\right)^*_{\mathbb{C}}, W_\bullet, F^\bullet\right)\right)$$

is a $\mathbb{Z}$-mixed Hodge structure and

$$0 \to \left(J_p/J_p^s\right)^* \to \left(J_p/J_p^{s+1}\right)^* \to K^s(X) \to 0$$

is an exact sequence of mixed Hodge structures.

2.2.2 Punctured Riemann Surface

Now let us consider the case of a punctured Riemann surface: $(S,p) = (X \setminus \{q\}, p)$ Notice that the first (co)homology group of $X \setminus \{q\}$ is isomorphic with the first (co)homology group of X. Remember that we denoted the augmentation ideal in the group ring $\mathbb{Z}\pi_1(X \setminus \{q\}, p)$ by J_{pq}. Our goal in this subsection is to construct a MHS on

$$\left(J_{pq}/J_{pq}^{s+1}\right)^*$$

such that the sequence

$$0 \to \left(J_{pq}/J_{pq}^{s}\right)^* \to \left(J_{pq}/J_{pq}^{s+1}\right)^* \to \bigotimes^s H^1(X) \to 0$$

becomes a short exact sequence of mixed Hodge structures. We proceed similarly as in Section 2.2.1.

As far as the complex of differential forms is concerned, things are more complicated as the following remark shows.

Remark 2.13 Removing one point from a compact Riemann surface does not change its singular cohomology. But nevertheless

$$\left(E_{\mathbb{R}}(X \setminus \{q\}),\ (E_{\mathbb{C}}(X \setminus \{q\}), F)\right)$$

is <u>not</u> an $\mathbb{R}$-Hodge complex. The filtration F does not even induce a Hodge structure on $H^1(E_{\mathbb{C}}(X \setminus \{q\}))$ if the genus of X is positive, because the following is true:[2]

$$F^1 H^1(E_{\mathbb{C}}(X \setminus \{q\})) = H^1(E_{\mathbb{C}}(X \setminus \{q\})).$$

Instead of the complex $E(X \setminus \{q\})$ we will use the complex of C^∞-forms on $X \setminus \{q\}$ with logarithmic singularities at q,

$$E^\bullet(X \log q) := \Gamma\left(\Omega^\bullet(X \log q) \otimes_{\Omega^0(X)} \mathcal{E}^{0,\bullet}(X)\right),$$

[2] A proof of this well-known identity can be given as follows (I learned this proof from Steenbrink) Denote by $\Omega^*(*q)$ the sheaf of holomorphic differential forms on $X \setminus \{q\}$, which are meromorphic on X Let $\mathcal{O}_X(*q) = \Omega^0(*q)$ Then $(\Omega^*(*q), \sigma_{\geq}) \hookrightarrow (E_{\mathbb{C}}(X \setminus \{q\}), F)$ induces a filtered morphism $(\mathbf{H}(\Omega^*(*q)), \sigma_{\geq}) \xrightarrow{\lambda} (H(E_{\mathbb{C}}(X \setminus \{q\}), F)$ The map $\mathbf{H}(\Omega^*(*q)) \to H(E_{\mathbb{C}}(X \setminus \{q\})$ is well-known to be an isomorphism (cf [GH78], p 453 (Grothendieck's algebraic de Rham Theorem)) Consider the short exact sequence of complexes of sheaves $0 \to \sigma_{\geq 1}\Omega^*(*q) \to \Omega^*(*q) \to [\mathcal{O}_X(*q) \to 0] \to 0$ and the corresponding long exact sequence of hypercohomology.

$$\cdot \to \mathbf{H}^1(\sigma_{\geq 1}\Omega^*(*q)) \to \mathbf{H}^1(\Omega^*(*q)) \to H^1(X, \mathcal{O}_X(*q)) \to \cdot\ .$$

Denote for any i by $\mathcal{O}(iq)$ the subsheaf of $\mathcal{O}(*q)$ of meromorphic functions with a pole of order at most i in q Since $\mathcal{O}(*q)/\mathcal{O}(iq)$ is concentrated in q, $H^1\left(X \setminus \{q\}, \mathcal{O}(*q)/\mathcal{O}(iq)\right) = 0$ and $H^1\left(X \setminus \{q\}, \mathcal{O}(iq)\right) \to H^1\left(X \setminus \{q\}, \mathcal{O}(*q)\right)$ is surjective For $i > 2g - 2$ we find by Riemann-Roch and Serre duality $H^1\left(X \setminus \{q\}, \mathcal{O}(iq)\right) = 0$ This shows $\sigma_{\geq 1}\mathbf{H}^1(\Omega^*(*q)) = \mathbf{H}^1(\Omega^*(*q))$, from which we conclude $F^1(H^1(E_{\mathbb{C}}(X \setminus \{q\}))) \supseteq \lambda\left(\sigma_{\geq 1}\mathbf{H}^1(\Omega^*(*q))\right) = \lambda\left(\mathbf{H}^1(\Omega^*(*q))\right) = H^1(E_{\mathbb{C}}(X \setminus \{q\}))$

in order to define a Hodge structure on $H^1(X \setminus \{q\})$ and $(J_{pq}/J_{pq}^{s+1})^*$. Note that $E^0(X \log q) = E^0(X)$.

The *weight filtration* $W_\bullet$ on $E^\bullet(X \log q)$ is multiplicative, i. e. $W_m \wedge W_n = W_{m+n}$, and hence determined by $W_{-1}E^\bullet(X \log q) = 0$ and $W_0 E^\bullet(X \log q) = E^\bullet(X)$ as well as $W_1 E^1(X \log q) = E^1(X \log q)$.

On $E^\bullet(X \log q)$ we have the *Hodge filtration*, which is defined by:

$$F^p E^\bullet(X \log q) := \bigoplus_{k \geq p} \Gamma\left(\Omega^k(X \log q) \otimes_{\Omega^0(X)} \mathcal{E}^{0,\bullet-k}(X)\right).$$

It is well-known that the differential d is strict with respect to $F^\bullet$ as $E^\bullet(X \log q)$ is part of a mixed Hodge complex. On the cohomology of this complex, the weight filtration is given by:

$$W_{l+m} H^m(E^\bullet(X \log q)) := \operatorname{im}\{H^m(W_l E^\bullet(X \log q)) \to H^m(E^\bullet(X \log q))\}.$$

Chen's theorem is valid for $E^\bullet(X \log q)$, since $E^\bullet(X \log q) \hookrightarrow E^\bullet_{\mathbb{C}}(X \setminus \{q\})$ is a quasi-isomorphism. That means, we know that the integration map defines an isomorphism

$$H^0 \bar{B}_s(E^\bullet(X \log q), p) \xrightarrow{\cong} \left(J_{pq}/J_{pq}^{s+1}\right)^*.$$

In particular, this isomorphism gives us an embedding of the lattice $(J_{pq}/J_{pq}^2)^* = H^1(X, \mathbb{Z}) \hookrightarrow H^1(E^\bullet(X \log q))$.

The inclusion $E^\bullet(X) \hookrightarrow E^\bullet(X \log q)$ induces an isomorphism $H^1(X) \cong H^1(X \setminus q)$ in cohomology, which maps $H^1(X, \mathbb{Z})$ to $H^1(X \setminus \{q\}, \mathbb{Z})$ and respects the filtrations $W_\bullet$ and $F^\bullet$. Therefore,

$$H^1(X \setminus \{q\}) = (H^1(X \setminus \{q\}, \mathbb{Z}), (H^1(X \setminus \{q\}, \mathbb{R}), W_\bullet), (H^1(X \setminus \{q\}, \mathbb{C}), W_\bullet, F^\bullet))$$

is a pure Hodge structure of weight 1, isomorphic with $H^1(X)$. We will identify $H^1(X)$ with $H^1(X \setminus \{q\})$ in this way.

Again similar to 2.2.1, the filtrations $W_\bullet$ and $F^\bullet$ on $E^\bullet(X \log q)$ induce filtrations $W_\bullet$ and $F^\bullet$ on $\bigoplus_{r=1}^s \bigotimes^r E^\bullet(X \log q)$ in a natural way by

$$W_l\left(\bigoplus_{r=1}^{s} \bigotimes^r E^\bullet(X \log q)\right) = \bigoplus_{r=1}^{s} \sum_{l_1+\cdots+l_r+r \leq l} W_{l_1}E^\bullet(X \log q) \otimes \cdots \otimes W_{l_r}E^\bullet(X \log q)$$

and

$$F^p\left(\bigoplus_{r=1}^{s} \bigotimes^r E^\bullet(X \log q)\right) = \bigoplus_{r=1}^{s} \sum_{p_1+\cdots+p_r \geq p} F^{p_1}E^\bullet(X \log q) \otimes \cdots \otimes F^{p_r}E^\bullet(X \log q).$$

Then define

$$W_l\left(H^0 \bar{B}_s(E^\bullet(X \log q), p)\right) := \operatorname{im}\left\{W_l \bar{\mathcal{K}}^s(E^\bullet(X \log q), p) \to H^0 \bar{B}_s(E^\bullet(X \log q), p)\right\}$$

and

$$F^p\left(H^0 \bar{B}_s(E^\bullet(X \log q), p)\right) := \operatorname{im}\left\{F^p \bar{\mathcal{K}}^s(E^\bullet(X \log q), p) \to H^0 \bar{B}_s(E^\bullet(X \log q), p)\right\}.$$

We want to prove the following theorem.

Theorem 2.14 *For all $s \geq 1$, the collection of data $H^0\bar{B}_s(X \setminus \{q\}, p) :=$*

$$\left(\left(H^0\bar{B}_s(E^\bullet_{\mathbb{R}}(X \log q), p), W_\bullet\right), \left(H^0\bar{B}_s(E^\bullet(X \log q), p), W_\bullet, F^\bullet\right) \right)$$

is an $\mathbb{R}$-mixed Hodge structure.

The proof of Theorem 2.14 is very similar to the proof of 2.4, once we have the following two propositions at our disposal. Therefore, we leave the proof itself to the reader.

The projection $\bigoplus_{r=1}^{s} \bigotimes^r E^1(X \log q) \to \bigotimes^s E^1(X \log q)$ induces the map R in the following commutative diagram.

$$\begin{tikzcd}
H^0\bar{B}_s(E^\bullet(X \log q), p) \arrow[r, "R"] \arrow[d] & \bigotimes^s H^1(X) \arrow[d, equal] \\
(J_{pq}/J_{pq}^{s+1})^* \arrow[r, "P"] & \bigotimes^s H^1(X)
\end{tikzcd}$$

Proposition 2.15 *The map*

$$R : \left(H^0\bar{B}_s(E^\bullet(X \log q), p), W_\bullet, F^\bullet\right) \to \left(\bigotimes^s H^1(X), W_\bullet, F^\bullet\right)$$

is surjective and strict with respect to the filtrations $W_\bullet$ and $F^\bullet$.

Proof: It is a consequence of Proposition 1.11 that R is surjective. Since the differential of $E^\bullet(X \log q)$ is strict with respect to the Hodge filtration, one can easily see from the same Proposition 1.11 that R is strict with respect to $F^\bullet$ and $W_\bullet$. □

Similar to Proposition 2.10 one can find a proof of the following proposition by applying Proposition 1.3.

Proposition 2.16 *For all s the sequence*

$$0 \to H^0\bar{B}_{s-1}(E^\bullet(X \log q), p) \to H^0\bar{B}_s(E^\bullet(X \log q), p) \xrightarrow{R} \bigotimes^s H^1(X) \to 0$$

is exact and strict with respect to $W_\bullet$ and $F^\bullet$. □

Remark 2.17 Together with Proposition 2.16 this theorem implies that the short exact sequence of 2.16 is an element of

$$\mathrm{Ext}_{\mathrm{MHS}}\left(\bigotimes^s H^1(X); H^0\bar{B}_{s-1}(E^\bullet(X \log q), p)\right).$$

In particular for the case $s = 2$ we get an element in:
$\mathrm{Ext}_{\mathrm{MHS}}\left(H^1(X)^{\otimes 2}, H^1(X)\right)$.

For $\mathbb{A} = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ define

$$\left(J_{pq}/J_{pq}^{s+1}\right)^*_{\mathbb{A}} := \mathrm{Hom}_{\mathbb{Z}}\left(J_{pq}/J_{pq}^{s+1}, \mathbb{A}\right).$$

By Chen's theorem, the filtrations $W_\bullet$ and $F^\bullet$ on $H^0\bar{B}_s(E^\bullet(X \log q), p)$ induce filtrations $W_\bullet$ and $F^\bullet$ on $\left(J_{pq}/J_{pq}^{s+1}\right)^*_{\mathbb{A}}$ for $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$ respectively.

Therefore, we can also summarize the results of this subsection by saying.

Theorem 2.18 *The collection of data*

$$\left(J_{pq}/J_{pq}^{s+1}\right)^* := \Big(\left(J_{pq}/J_{pq}^{s+1}\right)^*_{\mathbf{Z}};\ \left(\left(J_{pq}/J_{pq}^{s+1}\right)^*_{\mathbf{Q}},\ W_\bullet\right);\ \left(\left(J_{pq}/J_{pq}^{s+1}\right)^*_{\mathbf{C}},\ W_\bullet,\ F^\bullet\right)\Big)$$

is a $\mathbb{Z}$-mixed Hodge structure and

$$0 \to \left(J_{pq}/J_{pq}^{s}\right)^* \to \left(J_{pq}/J_{pq}^{s+1}\right)^* \to \bigotimes^s H^1(X) \to 0$$

is an exact sequence of mixed Hodge structures.

Chapter 3

Extensions and the Theta Divisor

In this and in the following chapter we want to study the sequence of mixed Hodge structures of Theorem 2.18 for the case $s = 2$. That means, we want to look at the element m_{pq} of $\mathrm{Ext}_{\mathrm{MHS}}\left(H^1(X) \otimes H^1(X),\, H^1(X)\right)$, which is represented by the sequence

$$m_{pq}: \qquad 0 \to H^1(X) \to \left(J_{pq}/J_{pq}^3\right)^* \xrightarrow{P} H^1(X) \otimes H^1(X) \to 0.$$

The leading question in these two chapters is the question about the relation between the extension m_{pq} and the sequence of mixed Hodge structures of Theorem 2.12 for the case $s = 2$, i. e. the element m_p of $\mathrm{Ext}_{\mathrm{MHS}}\left(K(X),\, H^1(X)\right)$, which is represented by

$$m_p: \qquad 0 \to H^1(X) \to \left(J_p/J_p^3\right)^* \to K(X) \to 0.$$

The problem: In this chapter we consider the following problem in particular.

Identify $H^2(X, \mathbb{Z})$ with $\mathbb{Z}$. Then there is a bilinear form

$$B : \left(H^1(X) \otimes H^1(X)\right) \times \left(H^1(X) \times H^1(X)\right) \longrightarrow \mathbb{Z},$$

given by $B\left((x_1 \otimes x_2), (y_1 \otimes y_2)\right) := (x_1 \cup y_2) \cdot (y_1 \cup x_2)$. B is non degenerate (B has mixed signature). Consider the subspace of $H^1(X) \otimes H^1(X)$, which is orthogonal to the kernel of the cup-product $K(X) \subset H^1(X) \otimes H^1(X)$ with respect to B. We will see that this subspace carries an induced $\mathbb{Z}$-MHS, isomorphic with $\mathbb{Z}(-1)$. We denote this MHS by $Q(X) \subset H^1(X) \otimes H^1(X)$. Since B is non degenerate we find over $\mathbb{Q}$:

$$K(X)_{\mathbb{Q}} \oplus Q(X)_{\mathbb{Q}} = H^1(X)_{\mathbb{Q}} \otimes H^1(X)_{\mathbb{Q}}. \tag{3.1}$$

Definition 3.1 Define $k_{pq} \in \mathrm{Ext}_{\mathrm{MHS}}\left(Q(X); H^1(X)\right)$ to be the restriction of m_{pq} to an extension of $Q(X)$ by $H^1(X)$:

$$k_{pq}: \qquad 0 \to H^1(X) \to P^{-1}\left(Q(X)\right) \xrightarrow{P} Q(X) \to 0.$$

Let us state two propositions, which we will prove later in Section 3.1.

Proposition 3.2 *The inclusion $K(X) \oplus Q(X) \hookrightarrow H^1(X) \otimes H^1(X)$ induces a natural exact sequence (write $H^1 = H^1(X)$, $K = K(X)$ and $Q = Q(X)$)*

$$0 \to \mathrm{Hom}_{\mathbb{Z}}\left((\frac{\mathbb{Z}}{2g\mathbb{Z}})^{2g}, \mathbb{Q}/\mathbb{Z} \right) \hookrightarrow \mathrm{Ext}_{\mathrm{MHS}}\left((H^1)^{\otimes 2}; H^1\right)$$
$$\xrightarrow{P_1 \oplus P_2} \mathrm{Ext}_{\mathrm{MHS}}\left(K; H^1\right) \oplus \mathrm{Ext}_{\mathrm{MHS}}\left(Q; H^1\right) \to 0,$$

where $P_1(m_{pq}) = m_p$ and $P_2(m_{pq}) = k_{pq}$.

Proposition 3.3 *There is a natural isomorphism*

$$\Psi : \mathrm{Ext}_{\mathrm{MHS}}\left(Q(X); H^1(X)\right) \xrightarrow{\cong} \mathrm{Pic}^0(X).$$

The question, that we want to answer in this chapter is the following.

$$\boxed{\text{What does } \Psi(k_{pq}) \in \mathrm{Pic}^0(X) \text{ look like?}}$$

The solution of this problem is given by the following theorem. The remaining part of this chapter is devoted to its proof.

Theorem 3.4

$$\Psi(k_{pq}) = (2g\, q - 2\, p - K) \quad \in \mathrm{Pic}^0(X),$$

where K is the canonical divisor in $\mathrm{Pic}^{2g-2}(X)$.

3.1 Preliminaries

Let us first describe $Q(X)$. Using the notation of 2.1 let $dx_1, \ldots, dx_{2g}$ be real harmonic C^∞ 1-forms representing dual cohomology classes of $[\gamma_1], \ldots, [\gamma_{2g}]$, i. e. $\int_{\gamma_i} dx_j = \delta_{ij}$. Then $Q(X) = \mathbb{Z}(\Xi)$, where Ξ is the generator of $Q(X)$:

$$\Xi := \sum_{\nu=1}^{g} \left([dx_\nu] \otimes [dx_{g+\nu}] - [dx_{g+\nu}] \otimes [dx_\nu]\right).$$

Consider the following commutative diagram of $\mathbb{Z}$-MHSs having exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K(X) & \to & K(X) \oplus Q(X) & \xrightarrow{\frac{1}{2g}\cup} & H^2(X) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow \cdot 2g & & \\
0 & \to & K(X) & \to & H^1(X) \otimes H^1(X) & \xrightarrow{\cup} & H^2(X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \frac{H^1(X) \otimes H^1(X)}{K(X) \oplus Q(X)} & \to & \frac{H^2(X)}{2g H^2(X)} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \qquad (3.2)$$

This diagram yields:

$$\frac{H^1(X)\otimes H^1(X)}{K(X)\oplus Q(X)} \cong \frac{H^2(X)}{2gH^2(X)}.$$

Now we can prove Proposition 3.2.

Proof of 3.2: The long exact sequence of the functor $\mathrm{Hom}_{\mathrm{MHS}}(\,\cdot\,;H^1)$ applied to the vertical exact sequence in the middle of (3.2) yields the exact sequence:

$$\mathrm{Hom}_{\mathrm{MHS}}\left(K\oplus Q;H^1\right) \to \mathrm{Ext}_{\mathrm{MHS}}\left(\frac{H^2}{2gH^2};H^1\right)$$
$$\to \mathrm{Ext}_{\mathrm{MHS}}\left(H^1\otimes H^1;H^1\right) \xrightarrow{\phi} \mathrm{Ext}_{\mathrm{MHS}}\left(K\oplus Q;H^1\right).$$

Since there is no morphism of pure Hodge structures of different weights in the category of MHSs, we have: $\mathrm{Hom}_{\mathrm{MHS}}\left(K\oplus Q;H^1\right)=0$. Observe moreover that $\mathrm{Ext}_{\mathrm{MHS}}\left(H^2/2gH^2;H^1\right)\cong\left(\mathbb{Z}/2g\mathbb{Z}\right)^{2g}$ (cf. [HS71], III.4).

Now we prove that ϕ is surjective. Given an extension $m \in \mathrm{Ext}_{\mathrm{MHS}}\left(K\oplus Q;H^1\right)$, say $m:\ 0\longrightarrow H^1\longrightarrow E\longrightarrow K\oplus Q\longrightarrow 0$, we construct an extension of MHSs: $m':\ 0\longrightarrow H^1\longrightarrow E'\longrightarrow H^1\otimes H^1\longrightarrow 0$ such that $\phi(m')=m$.

Let $L_{\mathbb{Z}}$ be either $H^1_{\mathbb{Z}}\otimes H^1_{\mathbb{Z}}$ or $K_{\mathbb{Z}}\oplus Q_{\mathbb{Z}}$, which are both free, then any short exact sequence $0\longrightarrow H^1_{\mathbb{Z}}\longrightarrow E_{\mathbb{Z}}\longrightarrow L_{\mathbb{Z}}\longrightarrow 0$ splits, i. e. $E_{\mathbb{Z}}\cong H^1_{\mathbb{Z}}\oplus L_{\mathbb{Z}}$. Therefore, there is an extension within the congruence class of m, whose lattice in the middle-term is $H^1_{\mathbb{Z}}\oplus(K_{\mathbb{Z}}\oplus Q_{\mathbb{Z}})$.

Then define m' by: $E'_{\mathbb{Z}}:=H^1_{\mathbb{Z}}\oplus(H^1_{\mathbb{Z}}\otimes H^1_{\mathbb{Z}})$, $E'_{\mathbb{Q}}:=E_{\mathbb{Q}}$ and $E'_{\mathbb{C}}:=E_{\mathbb{C}}$ and note that it is a preimage of m under ϕ. This shows that ϕ is surjective. □

Now we want to describe explicitly the identification of

$$\mathrm{Ext}_{\mathrm{MHS}}(Q(X);H^1(X))\cong\mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}(-1);H^1(X))$$

with the Jacobian $\mathrm{Jac}(X)$ resp. $\mathrm{Pic}^0(X)$ of X. We introduce some notation on our way.

Let $dz_1,\ldots,dz_g$ be a basis of $H^0(X,\Omega^1)\cong H^{1,0}(X)$ with period matrix:

$$\Omega=(\Omega_1,\Omega_2)=\left((\omega_{i,\nu})_{i,\nu},(\omega_{i,g+\nu})_{i,\nu}\right):=\left(\int_{\gamma_j}dz_i\right)_{\substack{i=1,\ldots,g\\ j=1,\ldots,2g}}$$

such that $\Omega_1=I\in\mathrm{Gl}(g,\mathbb{C})$. The classical Riemann period relations say that Ω_2 is symmetric with positive definite imaginary part. We represent the Jacobian of X as

$$\mathrm{Jac}(X):=\mathbb{C}^g/\Omega\mathbb{Z}^{2g}.$$

If we express $\sum_{\nu=1}^{g}(dx_\nu\otimes dx_{g+\nu}-dx_{g+\nu}\otimes dx_\nu)$ in terms of dz_j's, we find (by Riemann's period relations)

$$\sum_{\nu=1}^{g}(dx_\nu\otimes dx_{g+\nu}-dx_{g+\nu}\otimes dx_\nu)=\sum_{j,k=1}^{g}\left(a_{jk}dz_j\otimes d\bar z_k+\bar a_{jk}d\bar z_j\otimes dz_k\right)\quad(3.3)$$

with $A = (a_{jk})_{jk} = \left(\bar{\Omega}_2\Omega_1^t - \bar{\Omega}_1\Omega_2^t\right)^{-1}$. Observe that holds: $A^t = -\bar{A}$.

According to Carlson [Car80] there is an isomorphism

$$\mathrm{Ext}_{\mathrm{MHS}}\left(Q(X); H^1(X)\right) \cong \frac{\mathrm{Hom}\left(Q(X); H^1(X)\right)_{\mathbb{C}}}{F^0\,\mathrm{Hom}\left(Q(X); H^1(X)\right)_{\mathbb{C}} + \mathrm{Hom}\left(Q(X); H^1(X)\right)_{\mathbb{Z}}}$$

that maps an extension $0 \to H^1(X) \to E \to Q(X) \to 0$ to $[r_{\mathbb{Z}} \circ s_F]$, where $r_{\mathbb{Z}} : E_{\mathbb{Z}} \to H^1(X)_{\mathbb{Z}}$ is an integral retraction and $s_F : Q(X)_{\mathbb{C}} \to E_{\mathbb{C}}$ is a Hodge filtration preserving section. The proof of the following lemma is straightforward.

Lemma 3.5 *For $y^{(1)} = (y_1, \dots, y_g)$ and $y^{(2)} = (y_{g+1}, \dots, y_{2g})$ in $\mathbb{C}^g$ holds:*

$$\sum_{j=1}^{2g} y_j[dx_j] \in H^{1,0}(X) + H^1(X)_{\mathbb{Z}} \Leftrightarrow \Omega_1 y^{(2)} - \Omega_2 y^{(1)} \equiv 0 \mod \Omega\mathbb{Z}^{2g} \quad \square$$

Lemma 3.5 ensures that the following morphism is well-defined and that it is an isomorphism:

$$\begin{array}{cccc} \Phi: & \frac{\mathrm{Hom}\left(Q(X), H^1(X)\right)_{\mathbb{C}}}{F^0\,\mathrm{Hom}(Q(X),H^1(X))_{\mathbb{C}} + \mathrm{Hom}(Q(X),H^1(X))_{\mathbb{Z}}} & \longrightarrow & \mathbb{C}^g/\Omega\mathbb{Z}^{2g} \\ & \left\{[\psi] \text{ with } \psi(\Xi) = \sum_{i=1}^{2g} y_i dx_i\right\} & \longmapsto & \left(\Omega_1 y^{(2)} - \Omega_2 y^{(1)}\right), \end{array} \tag{3.4}$$

where $y^{(1)} := (y_1, \dots, y_g)$ and $y^{(2)} := (y_{g+1}, \dots, y_{2g})$.

Moreover we have the classical isomorphism, the Abel-Jacobi map:

$$\begin{array}{cccc} u: & \mathrm{Pic}^0(X) & \longrightarrow & \mathbb{C}^g/\Omega\mathbb{Z}^{2g} \\ & \sum_i (p_i - q_i) & \longmapsto & \left\{\sum_i \int_{q_i}^{p_i} dz_1, \dots, \sum_i \int_{q_i}^{p_i} dz_g\right\}. \end{array} \tag{3.5}$$

The composition of these isomorphisms gives an isomorphism

$$\Psi : \mathrm{Ext}_{\mathrm{MHS}}\left(Q(X); H^1(X)\right) \longrightarrow \mathrm{Pic}^0(X),$$

which does not depend on the choice of the basis $dz_1, \dots, dz_g$. This proves Proposition 3.3.

We can construct a Hodge filtration preserving section

$$s_F : Q(X) \longrightarrow H^0 \bar{B}_2\left(E(X \log q), p\right)$$

in the following way. Define

$$\mathfrak{X} := \frac{1}{2g} \sum_{\nu=1}^{g} \left(dx_\nu \wedge dx_{g+\nu} - dx_{g+\nu} \wedge dx_\nu\right).$$

Since by Riemann's bilinear relations holds $\mathfrak{X} \in F^1E^2(X)$ and because of the fact that $(E^\bullet(X), d)$ is strict with respect to the Hodge filtration, there is a $\mu_q \in F^1E\,(X \log q)$ such that holds:

$$2g\;\mathfrak{X} + d\mu_q = \sum_{\nu=1}^{g} (dx_\nu \wedge dx_{g+\nu} - dx_{g+\nu} \wedge dx_\nu) + d\mu_q = 0.$$

Therefore

$$s_F\,(\lambda\Xi) := \lambda \int \sum_{\nu=1}^{g} (dx_\nu dx_{g+\nu} - dx_{g+\nu} dx_\nu) + \mu_q \in H^0\bar{B}_2\,(E(X\log q), p)$$

defines a Hodge filtration preserving section from $Q(X)$ to $P^{-1}(Q(X))$.

Apart from this Hodge filtration preserving section we have an integral retraction:

$$\begin{array}{rcl} r_{\mathbb{Z}} : H^0\bar{B}_2\,(E(X\log q), p) & \longrightarrow & H^1(X) \\ \int I & \longmapsto & \sum_{j=1}^{2g}(\int_{\gamma_j} I)[dx_j], \end{array}$$

given by the basis $dx_1, \ldots, dx_{2g}$. Note that this retraction $r_{\mathbb{Z}}$ depends on the choice of the $[(\gamma_i - 1)] \in J_{pq}/J_{pq}^3$.

Having found s_F and $r_{\mathbb{Z}}$ we are ready to start our computation of the extension k_{pq}. Actually, we will compute $\Phi(r_{\mathbb{Z}} \circ s_F) \in \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$, the element in the Jacobian corresponding to the extension k_{pq}. For iterated integrals $\int I_1, \ldots, \int I_\alpha; \int J_1, \ldots, \int J_\beta \in \bar{B}_2\,(E(X\log q), p)$ we define the complex number:

$$\mathrm{Period}(\int I_1 \cdots \int I_\alpha; \int J_1 \cdots \int J_\beta) :=$$

$$\sum_{\nu=1}^{g} [\int_{\gamma_\nu} I_1 \cdots \int_{\gamma_\nu} I_\alpha][\int_{\gamma_{g+\nu}} J_1 \cdots \int_{\gamma_{g+\nu}} J_\beta] - [\int_{\gamma_{g+\nu}} I_1 \cdots \int_{\gamma_{g+\nu}} I_\alpha][\int_{\gamma_\nu} J_1 \cdots \int_{\gamma_\nu} J_\beta].$$

By the definition of $r_{\mathbb{Z}}$ and s_F we find:

$$r_{\mathbb{Z}} \circ s_F(\Xi) = \sum_{j=1}^{2g} \left(\int_{\gamma_j} \sum_{m=1}^{g} (dx_m dx_{g+m} - dx_{g+m} dx_m) + \mu_q \right) [dx_j].$$

Applying Φ to $r_{\mathbb{Z}} \circ s_F$ then yields:

$$\Phi(r_{\mathbb{Z}} \circ s_F) = \left(\mathrm{Period}(\int dz_i; \int \sum_{m=1}^{g} (dx_m dx_{g+m} - dx_{g+m} dx_m) + \mu_q) \right)_{i=1,\ldots,g}.$$

And by using formula (3.3) this becomes:

$$= \Bigg(\sum_{j,k=1}^{g} \left\{ a_{jk}\,\mathrm{Period}(\int dz_i; \int dz_j d\bar{z}_k) + \bar{a}_{jk}\,\mathrm{Period}(\int dz_i; \int d\bar{z}_j dz_k) \right\}$$
$$+ \mathrm{Period}(\int dz_i; \int \mu_q) \Bigg)_{i=1,\ldots,g}.$$

We will compute this expression in several steps.

3.2 Higher Reciprocity Law

Recall that we defined

$$\mathfrak{X} := \frac{1}{2g}\sum_{\nu=1}^{g}(dx_\nu \wedge dx_{g+\nu} - dx_{g+\nu}\wedge dx_\nu) \in F^1E^2(X)$$

and that we chose a $\mu_q \in F^1E\,(X \log q)$ such that

$$2g\,\mathfrak{X} + d\mu_q = 0.$$

In this section, we are going to prove the following *higher reciprocity law.*

Theorem 3.6 *For any holomorphic 1-form ω on X holds:*

$$\begin{aligned}\sum_{\nu=1}^{g}\{\int_{\gamma_\nu}\omega\int_{\gamma_{g+\nu}}\mu_q - \int_{\gamma_{g+\nu}}\omega\int_{\gamma_\nu}\mu_q\} \\ = 2g\int_p^q\omega + \sum_{j,k=1}^{g} 2a_{jk}\Big\{\mathrm{Period}(\int\omega;\int d\bar z_k dz_j) - \mathrm{Period}(\int\omega dz_j;\int d\bar z_k) \\ + \mathrm{Period}(\int\omega d\bar z_k;\int dz_j)\Big\}.\end{aligned}$$

The proof of this theorem will keep us occupied during the rest of this section and is a consequence of all lemmas, proved in this section. Since the formula is linear in ω, we may assume without loss of generality that $\omega = dz_i$ for some $i \in \{1,\dots,g\}$.

First, we study the form μ_q and compute its residue in q. We need the following well-known lemma.

Lemma 3.7 *Any $\eta \in E^1(X\log q)$, such that $d\eta \in E^2(X)$, can locally on a coordinate (U,z) on X around the point q be written as*

$$\eta = \mathrm{Res}_q\,\eta\cdot\frac{dz}{z} + \psi,$$

where ψ is a smooth 1-form in $E^1(U)$.

Proof: The Poincaré-Lemma tells us that there is a $\chi \in E^1(U)$ such that $d\chi = d\eta$. Hence, it suffices to prove Lemma 3.7 for a closed form η.

Write locally $\eta = h(z)\frac{dz}{z} + g(z)d\bar z$. Let $\rho := h(0)$. Then by the $\bar\partial$-Poincaré Lemma in One Variable ([GH78], p. 5) there is a smooth function $A : U \to \mathbb{C}$ such that $\frac{\partial A}{\partial \bar z} = \frac{\partial g}{\partial z}$. Write $\eta = \rho\frac{dz}{z} + \left(\frac{h(z)-h(0)}{z} - A(z)\right)dz + A(z)dz + g(z)d\bar z$. The function $z \mapsto \left(\frac{h(z)-h(0)}{z} - A(z)\right)$ is continuous in 0 with value $\frac{\partial h}{\partial z}(0) - A(0)$

and holomorphic (since $\bar\partial$ closed) everywhere else. By Riemann's extension theorem it is holomorphic on all of U. □

Concerning our form μ_q, we claim that there is a coordinate (U, z) on X around q such that

$$\boxed{\mu_q = \frac{2g}{2\pi i}\frac{dz}{z} + \varphi,} \tag{3.6}$$

where φ is a smooth 1-form in $E^1(U)$, i. e. we claim: $\operatorname{Res}_q \mu_q = \frac{2g}{2\pi i}$. This can be seen as follows. Let $\Delta_\varepsilon \subset U$ be the disk of radius ε around q in U. Then compute:

$$1 = \int_X \mathfrak{X} = \lim_{\varepsilon\to 0}\int_{X\setminus\Delta_\varepsilon} -\frac{1}{2g}\, d\mu_q = \lim_{\varepsilon\to 0}\int_{\partial\Delta_\varepsilon}\frac{1}{2g}\,\mu_q = \frac{2\pi i}{2g}\operatorname{Res}_q \mu_q.$$

Now we compute the expression Period($\int dz_i$; $\int \mu_q$) in two steps, the first of which is a computation on the universal covering space.

Let $\pi : \tilde X \longrightarrow X$ be the universal covering space of X. Fix a point $p_0 \in \pi^{-1}(p) \subset \tilde X$ and let $z_1, \ldots, z_g$ be functions on $\tilde X$ with $d\, z_i = \pi^* dz_i$ and $z_i(p_0) = 0$, $i = 1, \ldots, g$ (It will be clear from the context, whether $d\, z_i$ denotes dz_i or $\pi^* dz_i$). Let $\delta_1, \ldots, \delta_{2g}; \delta'_1, \ldots, \delta'_{2g}$ be a system of paths in $\tilde X$ with $\pi \circ \delta_i = \gamma_i$, $\pi \circ \delta'_i = \gamma_i^{-1}$, $i = 1, \ldots, 2g$, such that each of these paths starts at that point in $\tilde X$, where the path left to it in the sequence

$$\delta_1\delta_{g+1}\delta'_1\delta'_{g+1}\cdots\delta_g\delta_{2g}\delta'_g\delta'_{2g} \tag{3.7}$$

ends. Let the first path δ_1 start in p_0. Observe that because of the relation (2.2) in Section 2.1 the path defined by (3.7) is closed in $\tilde X$ and is a Jordan curve. Therefore it parametrizes the boundary ∂D of a closed subset $D \subset \tilde X$, which is homeomorphic to a closed disk in $\mathbb{R}^2$.

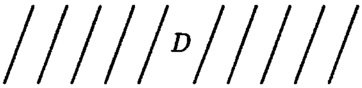
Lemma 3.8

$$\text{Period}\Big(\int dz_i; \int \mu_q\Big) = 2g\int_p^q dz_i + \sum_{j,k=1}^{g} 2a_{jk}\int_{\partial D} z_i \bar z_k dz_j$$

Proof: Comparing the values of z_i on δ'_ν with its values on δ_ν we observe for $i = 1, \ldots, g$:

$$\begin{aligned} z_i \circ \delta'_\nu - z_i \circ \delta_\nu &= \int_{\delta_{g+\nu}} \pi^* dz_i = \int_{\gamma_{g+\nu}} dz_i = \omega_{i,g+\nu} \\ z_i \circ \delta'_{g+\nu} - z_i \circ \delta_{g+\nu} &= \int_{\delta'_\nu} \pi^* dz_i = -\int_{\gamma_\nu} dz_i = -\omega_{i,\nu} \end{aligned}$$

as one can see by considering the picture:



$$\cdots \bullet \xrightarrow{\delta_\nu} \bullet \xrightarrow{\delta_{g+\nu}} \bullet \xrightarrow{\delta'_\nu} \bullet \xrightarrow{\delta'_{g+\nu}} \bullet \cdots \quad .$$

Let $\tilde{\mu}_q$ be the lifting $\pi^*\mu_q$ of μ_q onto $\tilde{X}$.

$$\int\limits_{\partial D} z_\imath \tilde{\mu}_q = \sum_{\nu=1}^{g} \int\limits_{\delta_\nu} z_\imath \tilde{\mu}_q + \int\limits_{\delta'_\nu} z_\imath \tilde{\mu}_q + \int\limits_{\delta_{g+\nu}} z_\imath \tilde{\mu}_q + \int\limits_{\delta'_{g+\nu}} z_\imath \tilde{\mu}_q$$

$$= \sum_{\nu=1}^{g} \int\limits_{\gamma_\nu} dz_\imath \int\limits_{\gamma_{g+\nu}} \mu_q - \int\limits_{\gamma_{g+\nu}} dz_\imath \int\limits_{\gamma_\nu} \mu_q = \text{Period}(\int dz_\imath; \int \mu_q).$$

Let q_0 be the unique point in D with $\pi(q_0) = q$ and let (U, z) be a coordinate around q such that $\mu_q = \frac{2g}{2\pi\imath}\frac{dz}{z} + \varphi$ with a smooth 1-form $\varphi \in F^1E^1(U)$. Moreover let S be a closed disk in U with $q_0 \in S \setminus \partial S$. Furthermore define the (1,0)-form

$$\psi := \sum_{\jmath,k=1}^{g} 2a_{\jmath k}\bar{z}_k dz_\jmath$$

on $\tilde{X}$ and observe $d\psi = \bar{\partial}\tilde{\mu}_q = d\tilde{\mu}_q$. Hence we can compute:

$$\text{Period}(\int dz_\imath; \int \mu_q) = \int\limits_{\partial D} z_\imath \tilde{\mu}_q = \int\limits_{\partial D} z_\imath\,(\tilde{\mu}_q - \psi) + z_\imath\psi$$

$$= \underbrace{\int\limits_{\partial(D\setminus S)} z_\imath\,(\tilde{\mu}_q - \psi)}_{=0 \text{ (Stokes)}} + \int\limits_{\partial S} \underbrace{z_\imath\,(\tilde{\mu}_q - \psi)}_{\text{meromorphic}} + \int\limits_{\partial D} z_\imath\psi$$

$$= 2\pi i\,\text{Res}_{q_0}\,[z_\imath\,(\tilde{\mu}_q - \psi)] + \int\limits_{\partial D} z_\imath\psi$$

$$= 2\pi i\,\text{Res}_{q_0}\big[z_\imath\big(\frac{2g}{2\pi i}\frac{dz}{z} + \underbrace{\varphi - \psi}_{\text{holomorphic}}\big)\big] + \int\limits_{\partial D} z_\imath\psi$$

$$= 2g\int\limits_{p}^{q} dz_\imath + \sum_{\jmath,k=1} 2a_{\jmath k} \int\limits_{\partial D} z_\imath \bar{z}_k dz_\jmath$$

This is the proof of the Lemma. □

Lemma 3.9 *For any i, j, $k \in \{1,\dots,g\}$ holds:*

$$\int_{\partial D} z_i \bar{z}_k dz_j$$
$$= \mathrm{Period}(\int dz_i;\ \int d\bar{z}_k dz_j) - \mathrm{Period}(\int dz_i dz_j;\ \int d\bar{z}_k) + \mathrm{Period}(\int dz_i d\bar{z}_k;\ \int dz_j).$$

Proof:

$$\int_{\partial D} z_i \bar{z}_k dz_j = \sum_{\nu=1}^{g} \int_{\delta_\nu} z_i \bar{z}_k dz_j + \int_{\delta'_\nu} z_i \bar{z}_k dz_j + \int_{\delta_{g+\nu}} z_i \bar{z}_k dz_j + \int_{\delta'_{g+\nu}} z_i \bar{z}_k dz_j$$

$$\begin{aligned}
&= \sum_{\nu=1}^{g} \int_{\delta_\nu} z_i \bar{z}_k dz_j - \int_{\delta_\nu} (z_i + \omega_{i,g+\nu})(\bar{z}_k + \bar{\omega}_{k,g+\nu}) dz_j \\
&\quad + \int_{\delta_{g+\nu}} z_i \bar{z}_k dz_j - \int_{\delta_{g+\nu}} (z_i - \omega_{i,\nu})(\bar{z}_k - \bar{\omega}_{k,\nu}) dz_j \\
&= \sum_{\nu=1}^{g} -\omega_{i,g+\nu} \int_{\delta_\nu} \bar{z}_k dz_j - \bar{\omega}_{k,g+\nu} \int_{\delta_\nu} z_i dz_j - \omega_{i,g+\nu}\bar{\omega}_{k,g+\nu}\omega_{j,\nu} \\
&\quad + \omega_{i,\nu} \int_{\delta_{g+\nu}} \bar{z}_k dz_j + \bar{\omega}_{k,\nu} \int_{\delta_{g+\nu}} z_i dz_j - \omega_{i,\nu}\bar{\omega}_{k,\nu}\omega_{j,g+\nu} \\
&= \sum_{\nu=1}^{g} -\omega_{i,g+\nu} \int_{\delta_\nu} \bar{z}_k dz_j - \bar{\omega}_{k,g+\nu} \int_{\delta_\nu} z_i dz_j - \omega_{i,g+\nu}\bar{\omega}_{k,g+\nu}\omega_{j,\nu} \\
&\quad + \omega_{i,\nu} \int_{\delta_{g+\nu}} (\bar{z}_k - \bar{\omega}_{k,\nu}) dz_j + \bar{\omega}_{k,\nu} \int_{\delta_{g+\nu}} (z_i - \omega_{i,\nu}) dz_j + \omega_{i,\nu}\bar{\omega}_{k,\nu}\omega_{j,g+\nu}
\end{aligned}$$

$$= \mathrm{Period}(\int dz_i;\ \int d\bar{z}_k dz_j) - \mathrm{Period}(\int dz_i dz_j;\ d\bar{z}_k) + \mathrm{Period}(\int dz_i d\bar{z}_k;\ dz_j)$$

□

Lemma 3.8 and Lemma 3.9 accomplish the proof of Theorem 3.6.

Concerning our computation of the extension k_{pq} we summarize: if we take into account that for any path γ, based at p, and any j, $k \in \{1,\dots,g\}$ holds

$$\int_\gamma dz_j d\bar{z}_k + \int_\gamma d\bar{z}_k dz_j = \int_\gamma dz_j \int_\gamma d\bar{z}_k,$$

then we get the following expression for $\Phi(r_{\mathbb{Z}} \circ s_F) \in \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$:

$$\begin{aligned}\Phi(r_{\mathbb{Z}} \circ s_F) = \Bigg(\sum_{j,k=1}^{g} a_{jk} \Big\{ & \operatorname{Period}(\int dz_i; \int dz_j \int d\bar{z}_k) \\ & +2 \operatorname{Period}(\int dz_i \int d\bar{z}_k; \int dz_j) \\ & -2 \operatorname{Period}(\int dz_i dz_j; d\bar{z}_k) \Big\} \\ & +2g \int_p^q dz_i \Bigg)_{i=1,\dots,g} \end{aligned} \tag{3.8}$$

Remark 3.10 In general, the following formula holds on a manifold. If φ, ψ are two 1-forms, β is a path based at a point p and u is another path from a point p' to p, then:

$$\int_{u\beta u^{-1}} \varphi\psi - \int_{\beta} \varphi\psi = \int_u \varphi \int_{\beta} \psi - \int_{\beta} \varphi \int_u \psi.$$

This formula allows to study the dependence of $\Phi(r_{\mathbb{Z}} \circ s_F)$ upon p. The dependence upon q is clear. Doing so, one obtains the following result.

Let $e_{pq} := \Psi^{-1}((p-q))$ be the extension corresponding to $(p-q) \in \operatorname{Pic}^0(X)$. Then the result is:

$$k_{pq} + 2g\, e_{pq} \quad \text{does not depend on } q,$$

$$k_{pq} + 2\, e_{pq} \quad \text{does not depend on } p.$$

Hence we may define

$$k_{qq} := \lim_{p \to q} k_{pq}.$$

This limit exists for the trivial reason:

$$\lim_{p\to q} k_{pq} = \lim_{p \to q} \{(k_{pq} + 2\, e_{pq}) - 2\, e_{pq}\} = k_{p'q} + 2g\, e_{p'q} \quad \forall p' \in X.$$

We find moreover

$$k_{pp} - k_{qq} = (2g-2)e_{pq}$$

This formula suggests the identity, which we will prove in the sequel,

$$\Psi(k_{pp}) = ((2g-2)p - K))$$

and proves it up to a constant. Finally, the expression for $\Phi(r_{\mathbb{Z}} \circ s_F)$ shows that we may restrict ourselves to the case $p = q$ in order to prove Theorem 3.4.

3.3 Higher Period Relation

In order to explain the way how we will prove *higher period relations*, we first give a short proof of the classical period relation: Riemann's first bilinear relation (cf. [Che77b] or [Gun69]).

Let φ, ψ be two holomorphic 1-forms on X. Because of $\varphi \wedge \psi = 0$, we know that $\int \varphi\psi$ is a homotopy functional. In particular it vanishes on the relation $\sum_{i=1}^{g} (c_i c_{g+i} - c_{g+i} c_i) \equiv 0 \mod J_p^3$ (cf. (2.3) on p. 29). Proposition 1.5 then gives:

$$\sum_{\nu=1}^{g} \int_{c_\nu} \varphi \int_{c_{g+\nu}} \psi - \int_{c_{g+\nu}} \varphi \int_{c_\nu} \psi = 0.$$

Also for iterated integrals of greater length one can derive comparable relations as we will show in the following. (Considerations of this type can also be found in [PY96]). Here, we want to apply this idea to continue our computation of k_{pq}. We will make use of the fact that

$$\sum_{j,k=1}^{g} a_{jk} \int dz_j\, dz_i\, dz_k \tag{3.9}$$

is a homotopy functional, since $dz_j \otimes dz_i \otimes dz_k$ is Chen-closed (use 1.13).

Observe that for elements in J_p, say $a = (\alpha - 1)$, $b = (\beta - 1)$, $c = (\gamma - 1)$ and $d = (\delta - 1)$, holds:

$$\begin{aligned} (\alpha\beta\gamma\delta - 1) &= (a+1)(b+1)(c+1)(d+1) - 1 \\ (\beta\alpha\gamma\delta - 1) &= (b+1)(a+1)(c+1)(d+1) - 1 \end{aligned}$$

and therefore

$$(\alpha\beta\gamma\delta - 1) - (\beta\alpha\gamma\delta - 1) \equiv (ab - ba) + (abc - bac) + (abd - bad) \mod J_p^4.$$

Moreover note that for $c' := (\gamma^{-1} - 1)$ holds: $abc' \equiv -abc \mod J_p^4$ and $ab(\gamma\delta - 1) \equiv abc + abd \mod J_p^4$. Let $e = (\varepsilon - 1)$ be an element in J_p^2. Applying the rules above to the following expression yields:

$$(\alpha\beta\alpha^{-1}\beta^{-1}\varepsilon - 1) \equiv (ab - ba) - (aba - baa) - (abb - bab) + e \mod J_p^4.$$

This shows that the relation (2.2) on page 29 yields the relation:

$$\begin{aligned} &\sum_{\nu=1}^{g} \{c_\nu c_{g+\nu} - c_{g+\nu} c_\nu \\ &- (c_\nu c_{g+\nu} c_\nu - c_{g+\nu} c_\nu c_\nu) - (c_\nu c_{g+\nu} c_{g+\nu} - c_{g+\nu} c_\nu c_{g+\nu})\} \equiv 0 \mod J_p^4. \end{aligned} \tag{3.10}$$

We obtain a higher period relation now, by applying the homotopy functional (3.9) to the equation (3.10). To give this higher period relation a form, which only involves iterated integrals of length 2, we will make use of the rules:

$$\int_{ab} dz_j dz_i dz_k = \int_a dz_j \int_b dz_i dz_k + \int_a dz_j dz_i \int_b dz_k$$

$$\int_a dz_i dz_j + \int_a dz_j dz_i = \int_a dz_i \int_a dz_j \text{ for all } a,\, b \in J_p.$$

In order to write this higher period relation down, we introduce some more notation. Recall that we have chosen $\Omega = (\Omega_1, \Omega_2) = (I, Z)$, where I is the $g \times g$ identity matrix and Z is symmetric with positive imaginary part $\Im(Z)$. Then $A = (\bar{Z} - Z)^{-1} = \frac{i}{2}\Im(Z)^{-1}$. Define for $i = 1, \dots, g$ the $g \times g$-matrices

$$I_1^i := \left(\int_{c_\nu} dz_i dz_j \right)_{\nu, j} \quad \text{and } I_2^i := \left(\int_{c_{g+\nu}} dz_i dz_j \right)_{\nu, j} \in \mathrm{Mat}(g \times g; \mathbb{C}).$$

Then we define the following two vectors with entries in $\mathrm{Mat}(g \times g; \mathbb{C})$

$$I_1 = \begin{pmatrix} I_1^1 \\ \vdots \\ I_1^g \end{pmatrix}, \quad I_2 = \begin{pmatrix} I_2^1 \\ \vdots \\ I_2^g \end{pmatrix} \in \mathrm{Mat}\left(g \times 1; \mathrm{Mat}(g \times g)\right).$$

For some matrix M, denote by $\operatorname{tr} M$ the *trace* of M and by $\operatorname{diag} M$ its *diagonal*. For a vector of matrices let *the trace* of this vector be the vector of the traces of the matrices.

The following theorem is the announced *higher period relation*. Its proof consists of calculations that follow the above recipe. We leave it to the reader.

Theorem 3.11

$$\begin{aligned} &(2\operatorname{tr}(I_2 A) - 2\operatorname{tr}(I_1 AZ)) + (\operatorname{diag}(ZAZ) - Z\operatorname{diag}(AZ)) \\ &+ (\operatorname{diag}(ZA) - Z\operatorname{diag}(A)) + (\operatorname{diag}(AZ) - Z\operatorname{diag}(ZA)) \equiv 0 \mod (I, Z)\mathbb{Z}^{2g}. \end{aligned}$$

□

Remark 3.12 Note that this expression could be shortened (e. g. $\operatorname{diag}(ZA) = \operatorname{diag}(AZ)$) but we consider this the right way to state the identity.

With the above notation, we use this higher period relation to continue our computation of the extension k_{pq} from section 3.2. Assume $p = q$. We had the following expression for $\Phi(r_{\mathbb{Z}} \circ s_F) \in \mathbb{C}^g / \Omega\mathbb{Z}^{2g}$:

$$\begin{aligned} \Phi(r_{\mathbb{Z}} \circ s_F) = \Bigg(\sum_{j,k=1}^{g} a_{jk} \sum_{\nu=1}^{g} \Big\{ &\omega_{i\nu}\omega_{j,g+\nu}\bar{\omega}_{k,g+\nu} - \omega_{i,g+\nu}\omega_{j\nu}\bar{\omega}_{k\nu} \\ &+2\,\omega_{i\nu}\omega_{j,g+\nu}\bar{\omega}_{k\nu} - 2\,\omega_{i,g+\nu}\omega_{j\nu}\bar{\omega}_{k,g+\nu} \\ &-2\, I^i_{1,\nu,j}\bar{\omega}_{g+\nu,k} + 2\, I^i_{2,\nu,j}\bar{\Omega}_{\nu,k} \Big\} \Bigg)_{i=1,\dots,g} \end{aligned}$$

$$= \Omega_1 \operatorname{diag}(\Omega_2^t A\bar{\Omega}_2) - \Omega_2 \operatorname{diag}(\Omega_1^t A\bar{\Omega}_1) + 2\Omega_1 \operatorname{diag}(\Omega_2^t A\bar{\Omega}_1) - 2\Omega_2 \operatorname{diag}(\Omega_1^t A\bar{\Omega}_2) - 2\operatorname{tr}(I_1 A\bar{\Omega}_2) + 2\operatorname{tr}(I_2 A\bar{\Omega}_1)$$

$$= \operatorname{diag}(ZA\bar{Z}) - Z\operatorname{diag}(A) + 2\operatorname{diag}(ZA) - 2Z\operatorname{diag}(A\bar{Z}) - 2\operatorname{tr}(I_1 A\bar{Z}) + 2\operatorname{tr}(I_2 A)$$

We can transform this expression such that it only contains (iterated) integrals over *holomorphic* forms. Observe

$$\operatorname{diag}(ZA\bar{Z}) = \operatorname{diag}(Z(\bar{Z}-Z)^{-1}(\bar{Z}-Z)) + \operatorname{diag}(ZAZ) = \operatorname{diag}(Z) + \operatorname{diag}(ZAZ)$$

and similarly:

$$2Z\operatorname{diag}(A\bar{Z}) \equiv 2Z\operatorname{diag}(AZ) \mod (I,Z)\mathbb{Z}^{2g}$$

and

$$2\operatorname{tr}(I_1 A\bar{Z}) = 2\operatorname{tr}(I_1) + 2\operatorname{tr}(I_1 AZ).$$

Using these identities we continue

$$\begin{aligned} \Phi(r_Z \circ s_F) \;\equiv\;& \operatorname{diag}(Z) + \operatorname{diag}(ZAZ) - Z\operatorname{diag}(A) \\ +\;& 2\operatorname{diag}(ZA) - 2Z\operatorname{diag}(AZ) \\ -\;& 2\operatorname{tr}(I_1) - 2\operatorname{tr}(I_1 AZ) + 2\operatorname{tr}(I_2 A) \mod (I,Z)\mathbb{Z}^{2g}. \end{aligned}$$

Notice: $\operatorname{diag}(ZA) - \operatorname{diag}(AZ) = 0$. When we apply Theorem 3.11, we finally get: $\Phi(r_Z \circ s_F) \equiv \operatorname{diag} Z - 2\operatorname{tr}(I_1) \mod (I,Z)\mathbb{Z}^{2g}$. The main result of this section hence is:

$$\boxed{\Phi(r_Z \circ s_F) \equiv \left(2\left[-\sum_{\nu=1}^{g}\int_{c_\nu} dz_i dz_\nu + \frac{1}{2}\int_{c_{g+i}} dz_i\right]\right)_{i=1,\dots,g} \mod (I,Z)\mathbb{Z}^{2g}.}$$

3.4 Riemann's Constant and the Theta Divisor

The expression

$$\left[-\sum_{\nu=1}^{g}\int_{c_\nu} dz_i dz_\nu + \frac{1}{2}\int_{c_{g+i}} dz_i\right]_{i=1,\dots,g}$$

played a role in mathematics already more than hundred and thirty years ago: it was used by Bernhard Riemann in 1865 ([Rie92], p. 213). He gave an expression of what nowadays is called *Riemann's constant* in terms of iterated integrals.[1] In this section we just want to briefly recall the context, in which he used it and want to finish the proof of Theorem 3.4.

[1] By the fact that already Riemann used iterated integrals we see that *iterated integrals* entered algebraic geometry in 1865 or earlier (in contrast to what is stated in [Hai87a], p 273)

Definition 3.13 The *vector of Riemann constants with respect to the basepoint* p is by definition:

$$\varkappa_p = (\varkappa_{p,i})_{i=1,\dots,g} := \left(-\sum_{\nu=1}^{g} \int_{c_\nu} dz_i dz_\nu + \frac{1}{2} \int_{c_{g+i}} dz_i \right)_{i=1,\dots,g} \in \mathbb{C}^g/(I,Z)\mathbb{Z}^{2g}.$$

Let

$$\theta(z) := \sum_{l \in \mathbb{Z}^g} e^{\pi i <l, Zl>} \cdot e^{2\pi i <l, z>}$$

be the *Riemann θ-function* of X on $\mathbb{C}^g$ with divisor $\Theta \subset \mathrm{Jac}(X)$. Denote by $X^{(d)}$ the d-fold symmetric product of X and by $W_{p,g-1}$ the image of the map (cf. (3.5)):

$$\begin{array}{ccc} X^{(g-1)} & \longrightarrow & \mathrm{Jac}\,(X) \\ \sum_{j=1}^{g-1} q_j & \longmapsto & \sum_{j=1}^{g-1} u(q_j - p). \end{array}$$

There is the following, most classical theorem of Riemann.

Theorem 3.14 (Riemann)

$$\Theta = W_{p,g-1} + \varkappa_p$$

Remark 3.15 Proofs of this theorem can be found in [Rie92] (VI, 22., pp. 132-136; XI, pp. 213-224) or [Lan02]. There are many proofs in modern language. We refer here to [Lew64], [Mum83] (Theorem 3.1, pp. 149-151) or to [GH78].

For the theory of θ-functions it is more convenient to define $\varkappa_p$ in $\mathbb{C}^g$ like [Rie92], [Lan02], [Lew64], [Fay73] (the vector of Riemann constants is defined here as $-\varkappa_p$) and [Mum83] do.

A consequence of this theorem and the theorem of Riemann-Roch is the following - also most classical - result.

Theorem 3.16 *K is the divisor of a holomorphic 1-form if and only if*

$$u(K - (2g-2)p) \equiv -2\varkappa_p \mod \Omega\mathbb{Z}^{2g}$$

Proof:[2] Assume K is the divisor of a differential form. By Riemann-Roch, if D is any effective divisor of degree $g-1$, then so is $K - D$. It follows that

$$W_{p,g-1} = u(K - (2g-2)p) - W_{p,g-1}.$$

Conversely, if K satisfies this identity, then it is the divisor of a differential form. We are done by observing: $\Theta = -\Theta$, which implies:

$$W_{p,g-1} = (W_{p,g-1} + \varkappa_p) - \varkappa_p = \Theta - \varkappa_p = -\Theta - \varkappa_p = -2\varkappa_p - W_{p,g-1} \quad \square$$

[2] This proof is due to [GH78], p. 340.

Thus, if we continue our computation of k_{pq}, then in the case $p = q$ we end up with:

$$\begin{array}{rrcl} & \Phi(r_{\mathbb{Z}} \circ s_f) & = & -u\,(K - (2g-2)p) \\ \Leftrightarrow & \Psi(k_{pp}) & = & (2g-2)p - K. \end{array}$$

And finally we derive from formula (3.8) of page 50 that if p and q are arbitrarily chosen points on X:

$$\boxed{\Psi(k_{pq}) = (2g \cdot q - 2 \cdot p - K) \quad \in \ \mathrm{Pic}^0(X)}$$

This proves Theorem 3.4. $\square$

Chapter 4

A Two-pointed Torelli Theorem

Also in this chapter we want to study the following extension of mixed Hodge structures (cf. Theorem 2.18) in $\mathrm{Ext}_{\mathrm{MHS}}\left(H^1(X)\otimes H^1(X),\, H^1(X)\right)$:

$$m_{pq}: \qquad 0 \to H^1(X) \to \left(J_{pq}/J_{pq}^3\right)^* \xrightarrow{P} H^1(X)\otimes H^1(X) \to 0$$

and compare it with the extension of MHSs (cf. Theorem 2.12), which is represented in $\mathrm{Ext}_{\mathrm{MHS}}\left(K(X),\, H^1(X)\right)$ by:

$$m_p: \qquad 0 \to H^1(X) \to \left(J_p/J_p^3\right)^* \to K(X) \to 0.$$

4.1 The Results

The following theorems of Hain and Pulte [Hai87c], [Pul88] show the relevance of the extension m_p.

Theorem 4.1 (Hain, Pulte) *The map*

$$\mathrm{Pic}^0 X \to \mathrm{Ext}_{\mathrm{MHS}}\left(K(X); H^1(X)\right),$$

which maps $(p-p')$ to $m_p - m_{p'}$ is well-defined and injective.

We write $(X,p) \cong (X,p')$ if there is an automorphism $\phi : X \to X$ that maps p to p'. For a point p on X we define *the set of alternatives for p* as

$$a_X(p) := \{p\} \cup \{p' \in X | m_{p'} = -m_p \text{ and } (X,p) \ncong (X,p')\}$$

The following is a consequence of Theorem 4.1.

Corollary 4.2 *$a_X(p)$ consists of at most two points. Up to automorphism of X, there cannot be more than one pair of points p and p' on X such that $a_X(p) = \{p,p'\} = a_X(p')$.*

Proof: The first assertion is an obvious consequence of 4.1. To prove the second assertion, assume that $\tilde{p}$ and $\tilde{p}'$ is another such pair with $a_X(\tilde{p}) = \{\tilde{p}, \tilde{p}'\} = a_X(\tilde{p})$. Then by 4.1, the divisors $p + p' = \tilde{p} + \tilde{p}'$ are linearly equivalent. It follows that either $\{p, p'\} = \{\tilde{p}, \tilde{p}'\}$ or X is hyperelliptic and the hyperelliptic involution maps $\{p, p'\}$ to $\{\tilde{p}, \tilde{p}'\}$. □

Together with the classical Torelli theorem, Hain and Pulte used Theorem 4.1 to prove the following *pointed Torelli theorem.*

Theorem 4.3 (Hain, Pulte) *Suppose that (X,p) and (Y,r) are two pointed compact Riemann surfaces. If there is a ring homomorphism*

$$\mathbb{Z}\pi_1(X,p)\Big/J_p(X)^3 \xrightarrow{\cong} \mathbb{Z}\pi_1(Y,r)\Big/J_r(Y)^3$$

which induces an isomorphism of MHSs, then there is an isomorphism $f : X \to Y$ with $f(p) \in a_Y(r)$.

Remark 4.4

- As far as the author knows, still no example is known of a pointed compact Riemann surface (X,p) with $|a_X(p)| = 2$.

- M. Pulte [Pul88] has shown that such an (X,p) with $|a_X(p)| = 2$ must have zero harmonic volume. B. Harris [Har83] proved that a generic smooth projective complex curve has non zero harmonic volume. Moreover, Pulte showed (loc. cit.) that, if there are two points p, p' with $a_X(p) = \{p,p'\} = a_X(p')$, then $(g-1)(p+p') - K = 0 \in \operatorname{Pic}^0 X$, where K is the canonical divisor.

- For pointed hyperelliptic curves (X,p) always holds: $a_X(p) = \{p\}$, since here $m_p = -m_{p'}$ implies $(X,p) \cong (X,p')$ by the hyperelliptic involution.

Our main result in this chapter is the following theorem.

Theorem 4.5 *For all $p \in X$, the map*

$$\operatorname{Pic}^0 X \to \operatorname{Ext}_{\mathrm{MHS}}\left(H^1(X) \otimes H^1(X); H^1(X)\right),$$

which maps $(q - q')$ to $m_{pq} - m_{pq'}$ is well-defined and injective.

We will find then by combining Theorem 4.5 with the results of Hain and Pulte the following Theorem.

Theorem 4.6 *The map*

$$\begin{array}{ccc} (X \times X) \setminus \Delta & \to & \operatorname{Ext}_{\mathrm{MHS}}\left(H^1(X) \otimes H^1(X); H^1(X)\right) \\ (p,q) & \mapsto & m_{pq}, \end{array}$$

is well-defined, extends to the diagonal Δ and is injective.

For a punctured Riemann surface $X \setminus \{q\}$ with basepoint p we define a sublattice P of rank 1 in $J_{pq}^2/J_{pq}^3 \cong \left(J_{pq}/J_{pq}^2 \otimes J_{pq}/J_{pq}^2\right)$ as follows. Observe that there is a natural isomorphism $J_{pq}/J_{pq}^2 \cong J_p/J_p^2$ and the ring structure on $\mathbb{Z}\pi_1(X,p)$ defines a map $\left(J_p/J_p^2 \otimes J_p/J_p^2\right) \to J_p^2/J_p^3$. Then let:

$$P_{pq}(X) := \ker\left\{J_{pq}^2/J_{pq}^3 \cong \left(J_{pq}/J_{pq}^2 \otimes J_{pq}/J_{pq}^2\right) \cong \left(J_p/J_p^2 \otimes J_p/J_p^2\right) \to J_p^2/J_p^3\right\}.$$

In Lemma 2.3 we saw already that holds:

$$P_{pq}(X) = \mathbb{Z}\left(\sum_{\nu=1}^{g}(c_\nu c_{g+\nu} - c_{g+\nu}c_\nu)\right).$$

And, it is clear that $P_{pq}(X)$ defines the cup-product

$$\cup : H^1(X) \otimes H^1(X) \to H^2(X)$$

by the formula (2.1.2): $[\varphi] \cup [\psi] = \left\{\sum_{\nu=1}^{g}\left(\int_{c_\nu}\varphi \int_{c_{g+\nu}}\psi - \int_{c_{g+\nu}}\varphi \int_{c_\nu}\psi\right)\right\}[X]$.

For points p and q on X we define

$$A_X(p,q) := \{(p,q)\} \cup \left\{ (p',q') \in X \times X \;\middle|\; \begin{array}{l} m_{p'q'} = -m_{pq} \quad \text{and} \\ \left(X \setminus \{q\}, p\right) \not\cong \left(X \setminus \{q'\}, p'\right) \end{array} \right\}.$$

The following is then a consequence of Theorem 4.6.

Corollary 4.7 *$A_X(p,q)$ consists of at most two elements.* □

Our results can then be applied to prove the following *two-pointed Torelli theorem.*

Theorem 4.8 *Suppose that $(X \setminus \{q\}, p)$ and $(Y \setminus \{s\}, r)$ are two punctured compact Riemann surfaces with basepoint. If there is a ring isomorphism*

$$\mathbb{Z}\pi_1(X \setminus \{q\}, p)\big/J_p(X \setminus \{q\})^3 \xrightarrow{\cong} \mathbb{Z}\pi_1(Y \setminus \{s\}, r)\big/J_r(Y \setminus \{s\})^3,$$

which induces an isomorphism of MHSs and that maps $P_{pq}(X)$ to $P_{rs}(Y)$, then there is a biholomorphism $f : X \to Y$ with $(f(p), f(q)) \in A_Y(r,s)$.

4.2 m_{pq} determines p and q

In this section we want to prove Theorem 4.5, that is, we want to prove that for all $p \in X$, the map

$$\mathrm{Pic}^0 X \to \mathrm{Ext}_{\mathrm{MHS}}\left(H^1(X) \otimes H^1(X); H^1(X)\right),$$

which maps $(q - q')$ to $m_{pq} - m_{pq'}$ is well-defined and injective.

This assertion follows from the following lemma. Let $H^1 = H^1(X)$.

Lemma 4.9 *For any element* $\sum_i (q_i - q_i') \in \mathrm{Pic}^0 X$ *holds:*

$$\sum_i (q_i - q_i') = 0 \in \mathrm{Pic}^0 X \;\Leftrightarrow\; \sum_i (m_{pq_i} - m_{pq_i'}) = 0 \in \mathrm{Ext}_{\mathrm{MHS}}\left((H^1)^{\otimes 2}; H^1\right).$$

Proof: Consider the isomorphism (cf. [Car80])

$$\mathrm{Ext}_{\mathrm{MHS}}\left(H^1 \otimes H^1; H^1\right) \xrightarrow{\cong} \frac{\mathrm{Hom}(H^1 \otimes H^1; H^1)_{\mathbf{C}}}{F^0 \mathrm{Hom}(H^1 \otimes H^1; H^1)_{\mathbf{C}} + \mathrm{Hom}(H^1 \otimes H^1; H^1)_{\mathbf{Z}}},$$

where the image of an extension m_{pq} is $[\phi_{pq}]$ for a certain $\phi_{pq} \in \mathrm{Hom}(H^1 \otimes H^1; H^1)_{\mathbf{C}}$. On an element $[\varphi] \otimes [\psi] \in H^1 \otimes H^1$, the homomorphism ϕ_{pq} is defined as follows.

If $[\varphi] \otimes [\psi] \in K(X)$, then choose a $\mu \in F^1 E^1(X)$ such that $\varphi \wedge \psi + d\mu = 0$. In this case we find

$$\phi_{pq}\left([\varphi] \otimes [\psi]\right) = \sum_{j=1}^{2g} \left(\int_{\gamma_j} \varphi\psi + \mu\right)[dx_j].$$

Note that this expression does not depend on q, hence we have for $q, q' \in X$ that $(\phi_{pq} - \phi_{pq'})_{|K(X)} \equiv 0$.

If $[\varphi] \otimes [\psi] \notin K(X)$ then choose a $\mu_q \in F^1 E^1(X \log q)$ such that $\varphi \wedge \psi + d\mu_q = 0$. Then

$$\phi_{pq}\left([\varphi] \otimes [\psi]\right) = \sum_{j=1}^{2g} \left(\int_{\gamma_j} \varphi\psi + \mu_q\right)[dx_j].$$

Therefore, we have for $q, q' \in X$ that

$$\left(\phi_{pq} - \phi_{pq'}\right)\left([\varphi] \otimes [\psi]\right) = \sum_{j=1}^{2g} \left(\int_{\gamma_j} \mu_q - \mu_{q'}\right)[dx_j].$$

Moreover, since $(\phi_{pq} - \phi_{pq'})$ is zero on $K(X)$, it is determined by its value on one element of $\left(H^1 \otimes H^1\right) \setminus K(X)$. For instance on $[dx_1] \otimes [dx_{g+1}]$.

Given the divisor $D = \sum_i (q_i - q_i')$, define the homomorphism

$$\Phi_D := \sum_i \left(\phi_{pq_i} - \phi_{pq_i'}\right) : H^1 \otimes H^1 \to H^1.$$

Now we will derive a series of equivalences. First, we have:

$$\begin{aligned} & \sum_i \left(m_{pq_i} - m_{pq_i'}\right) = 0 \in \mathrm{Ext}_{\mathrm{MHS}}\left(H^1 \otimes H^1; H^1\right) \\ \Leftrightarrow \quad & \Phi_D \in F^0 \mathrm{Hom}\left(H^1 \otimes H^1; H^1\right)_{\mathbf{C}} + \mathrm{Hom}\left(H^1 \otimes H^1; H^1\right)_{\mathbf{Z}}. \end{aligned}$$

Now let $\mathbf{w} \in H^{0,1} \otimes H^{0,1}$ be such that $[dx_1] \otimes [dx_{g+1}] - \mathbf{w} \in F^1(H^1 \otimes H^1) = H^{1,0} \otimes H^1 + H^1 \otimes H^{1,0}$. Note that $H^{0,1} \otimes H^{0,1} \subset K(X)$ and hence $\Phi_D(\mathbf{w}) = 0$.

Moreover $H^{1,0} \otimes H^{1,0} \subset K(X)$ and $\Phi_D\left(H^{1,0} \otimes H^{1,0}\right) = 0$. Therefore, we may continue the series of equivalences by:

$$\begin{aligned} &\Leftrightarrow \quad \Phi_D\left([dx_1] \otimes [dx_{g+1}] - \mathbf{w}\right) \in H^{1,0} + H^1_{\mathbb{Z}} \\ &\Leftrightarrow \quad \Phi_D\left([dx_1] \otimes [dx_{g+1}]\right) \in H^{1,0} + H^1_{\mathbb{Z}} \\ &\Leftrightarrow \quad \sum_{j=1}^{2g} \sum_i \left(\int_{\gamma_j} \mu_{q_i} - \mu_{q_i'}\right)[dx_j] \in H^{1,0} + H^1_{\mathbb{Z}} \end{aligned}$$

with $dx_1 \wedge dx_{g+1} + d\mu_{q_i} = 0$ for $\mu_{q_i} \in F^1E^1(X \log q_i)$ and $dx_1 \wedge dx_{g+1} + d\mu_{q_i'} = 0$ for $\mu_{q_i'} \in F^1E^1(X \log q_i')$. Note that this implies $\operatorname{Res}_{q_i} \mu_{q_i} = \frac{1}{2\pi i} = \operatorname{Res}_{q_i'} \mu_{q_i'}$ (cf. page 47). And by virtue of Lemma 3.5 we may go on:

$$\Leftrightarrow \quad \left(\text{Period}\left(\int dz_\nu;\ \int (\mu_{q_i} - \mu_{q_i'})\right)\right)_\nu \equiv 0 \mod \Omega\mathbb{Z}^{2g}.$$

By the reciprocity law for differentials of the third kind (cf. [GH78]), we find as $(\mu_{q_i} - \mu_{q_i'})$ is meromorphic with simple poles:

$$\begin{aligned} &\Leftrightarrow \quad \left(\sum_i \int_p^{q_i} dz_\nu\ 2\pi i \operatorname{Res}_{q_i}(\mu_{q_i} - \mu_{q_i'}) + \sum_i \int_p^{q_i'} dz_\nu\ 2\pi i \operatorname{Res}_{q_i'}(\mu_{q_i} - \mu_{q_i'})\right)_\nu \equiv 0 \\ &\qquad \mod \Omega\mathbb{Z}^{2g} \\ &\Leftrightarrow \quad \sum_i (q_i - q_i') = 0 \in \operatorname{Pic}^0 X. \end{aligned}$$

That proves the lemma. □

Our next job now is to prove Theorem 4.6. That is, the map

$$\begin{array}{ccc} (X \times X) \setminus \Delta & \to & \operatorname{Ext}_{\mathrm{MHS}}\left(H^1(X) \otimes H^1(X); H^1(X)\right) \\ (p, q) & \mapsto & m_{pq}, \end{array}$$

is well-defined, extends to the diagonal Δ and is injective.

Proof of 4.6: Note that the map of complex tori $\operatorname{Ext}_{\mathrm{MHS}}(H^1 \otimes H^1; H^1) \to \operatorname{Ext}_{\mathrm{MHS}}(K \oplus Q; H^1)$ is a covering map, since $\operatorname{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{Z}} \hookrightarrow \operatorname{Hom}(K \oplus Q; H^1)_{\mathbb{Z}}$. Moreover, we have the commutative diagram

$$\begin{array}{ccc} (X \times X) \setminus \Delta & \xrightarrow{\tilde{\varphi}} & \operatorname{Ext}_{\mathrm{MHS}}(H^1 \otimes H^1; H^1) \\ \big\downarrow & & \big\downarrow \text{covering map} \\ X \times X & \xrightarrow{\varphi} & \operatorname{Ext}_{\mathrm{MHS}}(K; H^1) \oplus \operatorname{Pic}^0 X \\ (p, q) & \longmapsto & \big(m_p,\ (2g\,q - 2p - K)\big). \end{array}$$

The map φ is continuous (m_p is – in a coordinate system – an expression of iterated integrals over paths with basepoint p). As the map $\tilde{\varphi}(p, q) = m_{pq}$ is a lifting of φ, we see that $\tilde{\varphi}$ is continuous too. The fact that the map $m_{pq} \mapsto (m_p, k_{pq})$ is a covering map tells us moreover that we may extend $\tilde{\varphi}$ to the diagonal Δ. Now by 4.1, the result of Hain and Pulte, the extension m_{pq} determines p. By Theorem 4.5 it determines also q. □

4.3 Proof of the Two-pointed Torelli Theorem

The last thing to do in this chapter is to prove Theorem 4.8, the two-pointed Torelli theorem. The proof goes along the lines of the proof of the pointed Torelli theorem in [Pul88].

Proof of 4.8: Let $J_{pq} = J_p(X \setminus \{q\})$ and $J_{rs} = J_r(Y \setminus \{s\})$. We have an isomorphism of MHSs, $\lambda : J_{pq}/J_{pq}^3 \xrightarrow{\cong} J_{rs}/J_{rs}^3$ and in particular, λ induces an isomorphism of Hodge structures

$$\lambda^* : H^1(Y) = W_1\left(J_{rs}/J_{rs}^3\right)^* \to W_1\left(J_{pq}/J_{pq}^3\right)^* = H^1(X),$$

which preserves the polarization. By the classical Torelli theorem (cf. for instance [Mar63]) we know that there is a biholomorphism $f : X \to Y$ such that $f^* : H^1(Y) \to H^1(X)$ is $\pm\lambda^*$. Since λ respects the *ring* structure, the by λ induced map

$$\left(J_{rs}^2/J_{rs}^3\right)^* \to \left(J_{pq}^2/J_{pq}^3\right)^*$$

is determined by $\lambda^* : H^1(Y) \to H^1(X)$ and hence,

$$f^* : \left(J_{rs}^2/J_{rs}^3\right)^* = H^1(Y) \otimes H^1(Y) \to H^1(X) \otimes H^1(X) = \left(J_{pq}^2/J_{pq}^3\right)^*$$

is equal to $\lambda^* \otimes \lambda^*$.

Without loss of generality, we may therefore assume that $(Y \setminus \{s\}, r) = (X \setminus \{q'\}, p')$ for two points p' and q' in X and that the following diagram commutes ($H^1 = H^1(X)$):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1 & \longrightarrow & \left(J_{pq}/J_{pq}^3\right)^* & \longrightarrow & H^1 \otimes H^1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pm id} & & \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & H^1 & \longrightarrow & \left(J_{p'q'}/J_{p'q'}^3\right)^* & \longrightarrow & H^1 \otimes H^1 & \longrightarrow & 0.
\end{array}$$

It follows that $m_{pq} = \pm m_{p'q'}$. This means that there either is an automorphism $\phi : (X \setminus \{q\}, p) \to (X \setminus \{q'\}, p')$ or $A_X(p,q) = \{(p,q);\ (p',q')\} = A_X(p',q')$. In both cases, the identity map is the map with the desired properties. □

Part II

A Mixed Hodge Structure on the Homotopy Data of an Irreducible Plane Curve Singularity

Chapter 5

The Nearby Fundamental Group

In this second part of the thesis we will give a construction of a MHS on the homotopy data of an irreducible plane curve singularity. This will be called *the MHS on the nearby fundamental group of the irreducible plane curve singularity.*

In this chapter we start out by first reviewing the language of local systems as we intend to use it. Then we describe the setting that we consider first and which arises, when considering irreducible plane curve singularities. We develop the notion of *nearby fundamental group* and show how these nearby fundamental groups form a local system. This local system is isomorphic with the corresponding *local system of fundamental groups of the fibers of the fibration.*

5.1 Local Systems

Here we want to introduce briefly the language of local systems, which we intend to use henceforth. We just recall the definitions and some basic facts. In the next sections we will define three local systems on different but homotopy equivalent spaces. We will conclude that certain homotopy equivalences between these spaces induce isomorphisms of the respective local systems just by comparing them at base points and considering the respective monodromy actions. The language to do that will be explained in this section. As references may serve [Spa66] p. 58 and [Mac71].

Definition 5.1 A *local system* on a topological space Y with values in a category C is a contravariant[1] functor from the fundamental groupoid $\Pi(Y)$ of Y to the category C.

[1]Here we multiply paths in their *natural order*, that is, $\alpha \star \beta$ means 'traverse α, then β'.

For any topological space Y and any category $\mathcal{C}$ there is a category of local systems on Y with values in $\mathcal{C}$. Morphisms of this category are defined to be natural transformations between local systems on Y with values in $\mathcal{C}$.

Two local systems on Y with values in $\mathcal{C}$ are called *isomorphic* if they are isomorphic in this category, i. e. when there is a natural equivalence between them.

Given a continuous map $f : Y \to Y'$, then f induces a contravariant functor f^* from the category of local systems on Y' with values in $\mathcal{C}$ to the category of local systems on Y with values in $\mathcal{C}$.

If f is a homotopy equivalence, then this functor is an equivalence of categories[2]. For local systems Γ' on Y' and Γ on Y we say: *f induces an isomorphism between Γ' and Γ*, iff $f^*\Gamma'$ is isomorphic with Γ.

On the other hand, if F is a covariant functor from $\mathcal{C}$ to some other category $\mathcal{D}$, then F induces a covariant functor from the category of local systems on Y with values in $\mathcal{C}$ to the category of local systems on Y with values in $\mathcal{D}$.

For an object A of a category $\mathcal{C}$ we denote by $\mathrm{Aut}(A)$ the group of automorphisms of A in $\mathcal{C}$. Given a group G, then we define *a representation* of G in $\mathcal{C}$ to be a homomorphism

$$G \longrightarrow \mathrm{Aut}(A) \qquad \text{for some object } A \text{ in } \mathcal{C}.$$

An isomorphism $\varphi : A \to B$ in $\mathcal{C}$ induces an isomorphism

$$\begin{aligned} c_\varphi : \mathrm{Aut}(A) &\longrightarrow \mathrm{Aut}(B) \\ \alpha &\longmapsto \varphi \circ \alpha \circ \varphi^{-1}. \end{aligned}$$

We call two representations $G \to \mathrm{Aut}(A)$ and $G \to \mathrm{Aut}(B)$ of G in $\mathcal{C}$ *conjugate* if there is an isomorphism $\varphi : A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Aut}(A) \arrow[rr, "c_\varphi"] & & \mathrm{Aut}(B) \\ & G \arrow[ul] \arrow[ur] & \end{tikzcd}$$

If Γ is a local system on Y and $y_0 \in Y$ a base point, then Γ induces a homomorphism, *the monodromy representation*

$$\Gamma_{y_0} : \pi_1(Y, y_0) \longrightarrow \mathrm{Aut}\, \Gamma(y_0).$$

For two local systems Γ and Γ' on Y, an isomorphism $\Phi \in \mathrm{Mor}(\Gamma, \Gamma')$, that is a natural equivalence from Γ to Γ', defines an isomorphism $\varphi : \Gamma(y_0) \longrightarrow \Gamma'(y_0)$

[2] Assume $g : Y' \to Y$ with $f \circ g \simeq id_{Y'}$ and $g \circ f \simeq id_Y$ and let $H : I \times Y \to Y$ be a homotopy between id_Y and $g \circ f$, then the natural equivalence between a local system Γ and $(g \circ f)^*\Gamma$ is given on $y \in Y$ by: $\Gamma(H(\cdot, y)) : \Gamma(y) \xrightarrow{\simeq} \Gamma(g \circ f(y))$.

such that

$$\begin{tikzcd} \mathrm{Aut}\,\Gamma(y_0) \arrow[rr, "c_\varphi"] & & \mathrm{Aut}\,\Gamma'(y_0) \\ & \pi_1(Y, y_0) \arrow[ul, "\Gamma_{y_0}"] \arrow[ur, "\Gamma'_{y_0}"'] & \end{tikzcd}$$

commutes. In this way, isomorphic local systems yield conjugate monodromy representations. But also the converse is true (cf. [Spa66], p. 58).

Proposition 5.2 *If Y is path-connected, two local systems Γ and Γ' on Y with values in C are equivalent if and only if the representations*

$$\Gamma_{y_0} : \pi_1(Y, y_0) \to \mathrm{Aut}\,\Gamma(y_0) \quad \textit{and} \quad \Gamma'_{y_0} : \pi_1(Y, y_0) \to \mathrm{Aut}\,\Gamma'(y_0)$$

are conjugate. □

Assume now that we are given two different but homotopy equivalent topological spaces with base points (Y, y_0) and $(Y', {y'}_0)$ and a local system Γ on Y as well as a local system Γ' on Y' – both with values in C. Let $f : Y \to Y'$ with $f(y_0) = y_0'$ be a homotopy equivalence between these topological spaces. Then the proposition allows us to conclude the following.

> *f induces an isomorphism between the local system Γ' and the local system Γ if and only if Γ and Γ' yield conjugate representations of $\pi_1(Y, y_0)$ resp. $\pi_1(Y', {y'}_0)$ in C, identified by means of f.*

5.2 The Setting

We first consider the following setting:
Assume we are given a map of space germs

$$h : (Z,\ D^+) \longrightarrow (\Delta,\ 0),$$

as well as tangent vectors $\vec{v} \in T_0\Delta$ and $\vec{w} \in T_{p_0}D_0$, where

- *Z is a complex manifold of dimension 2,*
- *$\Delta = \{z \in \mathbb{C} \big| |z| < 1\}$,*
- *$Z_0 := h^{-1}(0) = \bigcup_{i \geq 0} D_i$ is a connected reduced divisor with normal crossings (DNC) whose components are D_i with $i \geq 0$. They are such that $D^+ := \bigcup_{i>0} D_i$ is the union of closed compact Riemann surfaces D_i intersecting each other mutually in either one or no point. $D_0 \cap D^+$ consists of one point p_0 ($= p_{0l}$ for some l, which we assume to be 1, so: $p_0 = p_{01}$) and the respective space germs satisfy: $(Z \cap D_0,\ D^+ \cap D_0) = (D_0,\ p_0) \cong (\mathbb{C},\ 0)$.*

- *Define* $Z^* := Z \setminus h^{-1}(0)$ *and* $\Delta^* := \Delta \setminus \{0\}$ *as well as* $h^* := h_{|Z^*}$. *Then* $h^* : Z^* \longrightarrow \Delta^*$ *is a locally trivial* C^∞*-fibration, where the closure of each fiber (in a bigger representative of* (Z, D^+)*) is a compact Riemann surface with one boundary component.*

Let us define for $i \geq 0$

$$P_i := \bigcup_{j \neq i} D_i \cap D_j.$$

Frequently it will be convenient to use the notation " $[k < l]$" resp. " $[0 < k < l]$" for "all pairs k, l which satisfy $k < l$ and $D_k \cap D_l \neq \emptyset$" resp. "all pairs k, l which satisfy $0 < k < l$ and $D_k \cap D_l \neq \emptyset$".

We are going to use the notation of this setting throughout the whole second part of the thesis.

In chapter 8 we will extract such a situation like above with an additional action of finite cyclic groups on the fibration from the data of an irreducible plane curve singularity modulo right-equivalence.

5.3 Fundamental Group of the Nearby Fiber over a Tangent Vector

Let us explain in this section, what we mean by *the fundamental group of the nearby fiber over a tangent vector.*

5.3.1 The Hessian at a Double Point

Consider the Hessian of h in a double point $p_{kl} = D_k \cap D_l$. It is a symmetric bilinear form:

$$H(h) : T_{p_{kl}} Z \times T_{p_{kl}} Z \longrightarrow T_0 \Delta.$$

Since D_k and D_l intersect transversally we have $T_{p_{kl}} Z = T_{p_{kl}} D_k \oplus T_{p_{kl}} D_l$. Note that $H(h)(T_{p_{ij}} D_i \times T_{p_{ij}} D_i) = 0$ for $\{i, j\} = \{k, l\}$, that is to say that $T_{p_{kl}} D_k$ and $T_{p_{kl}} D_l$ are isotropic subspaces of $H(h)$ in $T_{p_{kl}} Z$. Therefore, the following restriction of the Hessian determines it completely:

$$\eta_{kl} : T_{p_{kl}} D_k \times T_{p_{kl}} D_l \longrightarrow T_0 \Delta.$$

Putting all such maps for all double points together, we obtain a map:

$$\langle \cdot, \cdot \rangle : \coprod_{[k<l]} T_{p_{kl}} D_k \times T_{p_{kl}} D_l \longrightarrow T_0 \Delta.$$

When we are given local coordinates (x, y) around a double point p_{kl} in Z and a coordinate t on Δ such that in these coordinates

$$h(x, y) = xy,$$

then we find

$$\left\langle a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}\right\rangle = a \cdot b\, \frac{\partial}{\partial t}.$$

5.3.2 Paths in Z_0 over $\vec{v}$

Let $\vec{v}$ be a tangent vector in $T_0\Delta$. We define a *path in* Z_0 *over* $\vec{v}$ or a *path in* $Z_{\vec{v}}$ to be a continuous map

$$\gamma : [a, b] \longrightarrow D^+ \subset Z_0,$$

which, when considered as map to Z, is piecewise smooth and which satisfies the following conditions:

(i) γ is right- and left differentiable at all double points p_{kl}, $[k < l]$.

(ii) For each double point p_{kl} with $[0 < k < l]$ there are two possibilities:

  (a) either γ stays in the same component while passing p_{kl}, or

  (b) it changes from one component to another.

Assume that $\tau_0 \in [a, b]$ is such that $\gamma(\tau_0) = p_{kl}$. If γ stays in the same component we require (and also for p_0)

$$-\dot{\gamma}^{\leq\tau_0}(\tau_0) = \dot{\gamma}_{\geq\tau_0}(\tau_0) \neq 0$$

and if γ changes from one component to another we impose

$$\left\langle -\dot{\gamma}^{\leq\tau_0}(\tau_0),\ \dot{\gamma}_{\geq\tau_0}(\tau_0)\right\rangle = \vec{v},$$

where $\gamma^{\leq\tau_0} : [a; \tau_0] \to Z_0$ and $\gamma_{\geq\tau_0} : [\tau_0; b] \to Z_0$ denote the respective restrictions of γ. If τ_0 is a resp. b, then we define $\dot{\gamma}^{\leq a}(a) := \dot{\gamma}^{\leq b}(b)$ resp. $\dot{\gamma}_{\geq b}(b) := \dot{\gamma}_{\geq a}(a)$.

5.3.3 Closed Paths in Z_0 over $\vec{v}$ based at $\vec{w}$

Let $\vec{w}$ be a tangent vector in $T_{p_0}D_0$. We define *a path over* $\vec{v}$ *based at* $\vec{w}$ to be a path $\gamma : [a, b] \to Z_0$ over $\vec{v}$ with $\gamma(a) = \gamma(b) = p_0$ such that holds:

$$\langle \vec{w}, \dot{\gamma}_{\geq a}(a)\rangle = \vec{v} \quad \text{and} \quad \langle -\dot{\gamma}^{\leq b}(b), \vec{w}\rangle = \vec{v}.$$

It follows that: $\dot{\gamma}_{\geq a}(a) = -\dot{\gamma}^{\leq b}(b)$.

5.3.4 Homotopy of Paths in Z_0 over $\vec{v}$ Relative to their Endpoints

Let p and q be two points in $D^+ \subset Z_0$ and let γ_0 and γ_1 be two paths over $\vec{v}$ with the endpoints p and q, i. e. $\gamma_0(0) = \gamma_1(0) = p$ and $\gamma_0(1) = \gamma_1(1) = q$. We say that there is a *homotopy over $\vec{v}$ relative to their endpoints p and q* between γ_0 and γ_1, if there are continuous maps a, $b : I \to \mathbb{R}$ with $a(s) < b(s)$ for all $s \in I$ and

$$H : \{(t,s) | s \in I,\ t \in [a(s), b(s)]\} \longrightarrow Z_0$$

such that $H(\cdot, 0) = \gamma_0$ and $H(\cdot, 1) = \gamma_1$ as well as that for any $s \in I$ the map $H(\cdot, s) : [a(s), b(s)] \to Z_0$ is a path over $\vec{v}$ with endpoints p and q. If the context is clear we will call two paths in Z_0 over $\vec{v}$ with the same endpoints *nearby homotopic relative to their endpoints*, if there is a homotopy over $\vec{v}$ relative to their endpoints between them. Note that the notion of nearby homotopy relative to the endpoints p and q as defined above yields an equivalence relation on the set of all paths in Z_0 over $\vec{v}$ with endpoints p and q.

5.3.5 Homotopy of Paths in Z_0 over $\vec{v}$ based at $\vec{w}$

We say that there is a *homotopy over $\vec{v}$ based at $\vec{w}$* between two paths γ_0 and γ_1 in Z_0 over $\vec{v}$ based at $\vec{w}$, if there are continuous maps a, $b : I \to \mathbb{R}$ with $a(s) < b(s)$ for all $s \in I$ and

$$H : \{(t,s) | s \in I,\ t \in [a(s), b(s)]\} \longrightarrow Z_0$$

such that $H(\cdot, 0) = \gamma_0$ and $H(\cdot, 1) = \gamma_1$ as well as that for any $s \in I$ the map $H(\cdot, s) : [a(s), b(s)] \to Z_0$ is a path over $\vec{v}$ based at $\vec{w}$. If there is no chance of confusion, we will call two paths in Z_0 over $\vec{v}$ based at $\vec{w}$ *nearby homotopic*, if there is a homotopy over $\vec{v}$ based at $\vec{w}$ between them. Note that the notion of *nearby homotopy* as defined above gives an equivalence relation on the set of all paths in Z_0 over $\vec{v}$ based at $\vec{w}$.

Moreover, remark that a homotopy over $\vec{v}$ based at $\vec{w}$ between two paths over $\vec{v}$ based at $\vec{w}$ is in particular a homotopy over $\vec{v}$ relative to the basepoint p_0 between these paths.

Proposition 5.3

(i) Every path $\gamma : [a, b] \to Z_0$ over $\vec{v}$ (based at $\vec{w}$) is nearby homotopic relative to its endpoints (resp. based at $\vec{w}$) to a path $\gamma_I : I \to Z_0$ over $\vec{v}$ (based at $\vec{w}$).

(ii) Assume, two paths over $\vec{v}$, say $\gamma_0, \gamma_1 : I \to Z_0$ with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ (resp. both based at $\vec{w}$), are nearby homotopic relative to their endpoints (resp. based at $\vec{w}$). Then there is a nearby homotopy

$$H : I \times I \longrightarrow Z_0$$

such that $H(\cdot, 0) = \gamma_0$ and $H(\cdot, 1) = \gamma_1$ as well as $H(\cdot, s)$ is a path over $\vec{v}$ with the endpoints $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ (resp. based at $\vec{w}$) for all $s \in I$.

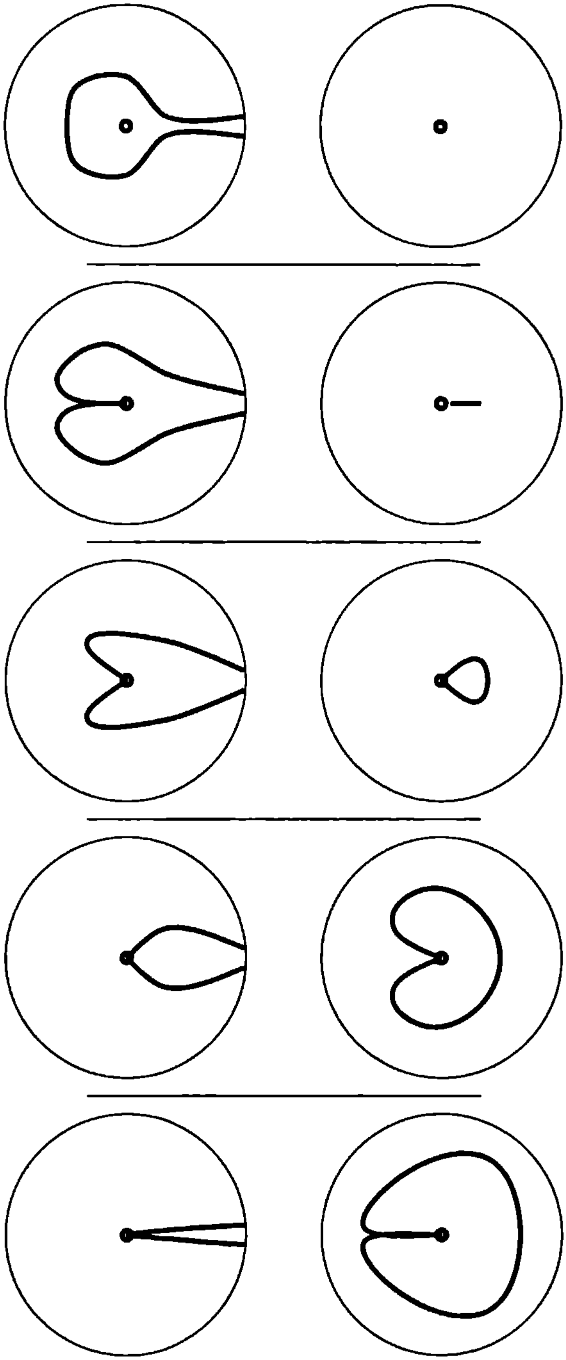


Figure 5.1: An example of a homotopy over $\vec{v}$ in the x-coordinate (to the left) and the y-coordinate (to the right).

Proof:
ad (i): Let $\gamma : [a,b] \to Z_0$ be a path over $\vec{v}$ based at $\vec{w}$. Define a, $b : I \to \mathbb{R}$ by $a(s) := a \cdot s$ and $b(s) := b \cdot s + (1-s)$ and observe that $[a(0), b(0)] = I$ and $[a(1), b(1)] = [a,b]$. Moreover we define a map

$$A : \left\{(t,s) \middle| s \in I,\, t \in [a(s), b(s)]\right\} \longrightarrow [a,b]$$
$$(t,s) \longmapsto \left(a + \tfrac{t-a(s)}{b(s)-a(s)}(b-a)\right)$$

and consider the composition with γ, which we call $\tilde{H}$:

$$\tilde{H} : \left\{(t,s) \middle| s \in I,\, t \in [a(s), b(s)]\right\} \longrightarrow Z_0.$$

Note that $\tilde{H}(\cdot, s) : [a(s), b(s)] \to Z_0$ is a path over $\frac{(b-a)^2}{(b(s)-a(s))^2} \cdot \vec{v}$ based at $\frac{(b-a)}{(b(s)-a(s))} \cdot \vec{w}$.

Choose a coordinate t on Δ such that $\vec{v} = \frac{\partial}{\partial t}$ and choose around each double point p_{kl} local coordinates $(x,y) : W_{kl} = U_{kl}^k \times U_{kl}^l \to \mathbb{C}^2$ such that the function h in the coordinates (x,y) and t looks like: $h(x,y) = x \cdot y$. For the double point p_0 we may assume that y is the coordinate on D_0 and that $\vec{w} = \frac{\partial}{\partial y}$.

Let $\sigma : \mathbb{R}^{\geq 0} \to [0;1]$ be a bump function, which is $\equiv 1$ near 0 and $\equiv 0$ on $\mathbb{R}^{\geq r_0}$ for a sufficiently small $r_0 > 0$ and strictly decreasing inbetween.

Then modify $\tilde{H}$ to a map $H : \left\{(t,s) \middle| s \in I,\, t \in [a(s), b(s)]\right\} \to Z_0$ by changing it on the sets $\tilde{H}^{-1}(W_{kl}) \subset \left\{(t,s) \middle| s \in I,\, t \in [a(s), b(s)]\right\}$ as follows. For $(t,s) \in \tilde{H}^{-1}(W_{kl})$ we set:

$$x \circ H(t,s) := x \circ \tilde{H}(t,s) \left(\sigma\left(|\tilde{H}(t,s)|\right) \frac{b(s)-a(s)}{b-a} + \left(1 - \sigma\left(|\tilde{H}(t,s)|\right)\right)\right)$$

and

$$y \circ H(t,s) := y \circ \tilde{H}(t,s) \left(\sigma\left(|\tilde{H}(t,s)|\right) \frac{b(s)-a(s)}{b-a} + \left(1 - \sigma\left(|\tilde{H}(t,s)|\right)\right)\right).$$

This new map on $\tilde{H}^{-1}(W_{kl})$ coincides with the map $\tilde{H}$ on

$$\tilde{H}^{-1}\left(\left\{(x,y) \in W_{kl} \middle| |x| > r_0,\, |y| > r_0\right\}\right).$$

Therefore, we defined a continuous map

$$H : \left\{(t,s) \middle| s \in I,\, t \in [a(s), b(s)]\right\} \longrightarrow Z_0$$

such that $H(\cdot, 1) = \gamma$ and $H(\cdot, s) : [a(s), b(s)] \to Z_0$ is a path over $\vec{v}$ based at $\vec{w}$. In particular, we proved that γ is nearby homotopic to $H(\cdot, 0) : I \to Z_0$.
ad (ii): Let γ_0, $\gamma_1 : I \to Z_0$ be nearby homotopic. Then there are functions a, $b : I \to \mathbb{R}$ with $a(s) < b(s)$ for all $s \in I$ and $[a(0), b(0)] = [a(1), b(1)] = I$ and there exists a nearby homotopy

$$H : \left\{(t,s) \middle| s \in I,\, t \in [a(s), b(s)]\right\} \longrightarrow Z_0$$

such that $H(\cdot,0) = \gamma_0$ and $H(\cdot,1) = \gamma_1$ as well as that for any $s \in I$ the map $H(\cdot,s) : [a(s), b(s)] \to Z_0$ is a path over $\vec{v}$ based at $\vec{w}$.

Consider the map

$$\begin{aligned} A : I \times I &\longrightarrow \{(t,s) | s \in I,\, t \in [a(s), b(s)]\} \\ (\tau, s) &\longmapsto (a(s) + \tau\,(b(s) - a(s))\,,\, s) \end{aligned}$$

and the composition with $\tilde{H}$, which we call $\tilde{\tilde{H}}$:

$$\tilde{\tilde{H}} := \tilde{H} \circ A : I \times I \longrightarrow Z_0.$$

Note that for $s \in I$ the map $\tilde{\tilde{H}}(\cdot,s) : I \to Z_0$ is a path over $(b(s) - a(s))^2\vec{v}$ based at $(b(s) - a(s))\vec{w}$ and that in particular $\tilde{\tilde{H}}(\cdot,0) = \gamma_0$ and $\tilde{\tilde{H}}(\cdot,1) = \gamma_1$. Similarly as in (i) we can deform $\tilde{\tilde{H}}$ to a nearby homotopy

$$H : I \times I \longrightarrow Z_0$$

between γ_0 and γ_1 by multiplying x- and y-coordinate of a point $\tilde{\tilde{H}}(t,s)$ near a double point p_{kl} with

$$\left(\frac{1}{b(s) - a(s)}\right) \cdot \sigma\left(|\tilde{\tilde{H}}(t,s)|\right) + \left(1 - \sigma\left(|\tilde{\tilde{H}}(t,s)|\right)\right).$$

□

Remark: Observe that, for example, each γ for which there is a $\lambda \in [0, \frac{1}{2}[$ such that

$$\gamma(\frac{1}{2} - \tau) = \gamma(\frac{1}{2} + \tau) \qquad \forall\, \tau \in [0; \lambda]$$

is *homotopic over $\vec{v}$ based at $\vec{w}$* with the path that does not go all the way to $\gamma(\frac{1}{2})$ but returns already from $\gamma(\frac{1}{2} - \lambda) = \gamma(\frac{1}{2} + \lambda)$, that is the path

$$\begin{aligned} \tilde{\gamma} : [0; 1 - 2\lambda] &\longrightarrow Z_0 \\ t &\longmapsto \begin{cases} \gamma(t) & \text{for } t \leq \frac{1}{2} - \lambda \\ \gamma(t + 2\lambda) & \text{for } t \geq \frac{1}{2} - \lambda \end{cases} . \end{aligned}$$

Let $a(s) \equiv 0$ and $b(s) = 1 - 2s\lambda$. Then the homotopy is given by

$$\begin{aligned} H : \{(t,s) | s \in I,\, t \in [a(s), b(s)]\} &\longrightarrow Z_0 \\ t &\longmapsto \begin{cases} \gamma(t) & \text{for } t \leq \frac{1}{2} - s\lambda \\ \gamma(t + 2s\lambda) & \text{for } t \geq \frac{1}{2} - s\lambda \end{cases} \end{aligned}$$

We denote the set of all 'homotopy over $\vec{v}$ based at $\vec{w}$'-classes of paths in Z_0 over $\vec{v}$ based at $\vec{w}$ by:

$$\boxed{\pi_1(Z_{\vec{v}}, \vec{w})}.$$

Again, if the context is clear we will refer to elements of this set as to *nearby homotopy classes.* $\pi_1(Z_{\vec{v}}, \vec{w})$ itself will be called *the nearby fundamental group (over $\vec{v}$ based at $\vec{w}$).* The term 'group' is justified by the following proposition.

Proposition 5.4 *$\pi_1(Z_{\vec{v}}, \vec{w})$ with the composition of 'paths in Z_0 over $\vec{v}$ based at $\vec{w}$' is a group.*

Proof: Note first that the composition of two paths in Z_0 over $\vec{v}$ based at $\vec{w}$ is again a path in Z_0 over $\vec{v}$ based at $\vec{w}$.

Furthermore, the composition of such paths gives a well-defined operation on the set of homotopy classes, which is associative.

All paths $\gamma : I \to Z_0$ over $\vec{v}$ based at $\vec{w}$ with $\gamma(\frac{1}{2} - \tau) = \gamma(\frac{1}{2} + \tau)$ for all $\tau \in [0; \frac{1}{2}]$ are mutually nearby homotopic. Call their nearby homotopy class e and note that it is the neutral element with respect to the above defined operation on homotopy classes.

Given an element $[\gamma] \in \pi_1(Z_{\vec{v}}, \vec{w})$, the inverse is represented by the path $t \mapsto \gamma(1-t)$ for $t \in I$. □

5.4 The Local System of Nearby Fundamental Groups

The association of $\pi_1(Z_{\vec{v}}, \vec{w})$ to a pair of non zero tangent vectors $\vec{v} \in (T_0\Delta)^* := T_0\Delta \setminus \{0\}$ and $\vec{w} \in (T_{p_0}D_0)^* := T_{p_0}D_0 \setminus \{0\}$ gives rise to a local system of groups on $(T_0\Delta)^* \times (T_{p_0}D_0)^*$ as we want to point out in the sequel. We call this local system *the local system of nearby fundamental groups* and denote it by:

$$\boxed{\{\pi_1(Z_{\vec{v}}, \vec{w})\}_{(\vec{v},\vec{w})\in(T_0\Delta)^*\times(T_{p_0}D_0)^*}}.$$

For a homotopy class modulo endpoints represented by a path

$$\vec{\eta} = \vec{\alpha} \times \vec{\beta} : I \to (T_0\Delta)^* \times (T_{p_0}D_0)^*,$$

i. e. a morphism of the fundamental groupoid of $(T_0\Delta)^* \times (T_{p_0}D_0)^*$, we define

$$\vec{\eta}_* : \pi_1(Z_{\vec{\alpha}(0)}, \vec{\beta}(0)) \longrightarrow \pi_1(Z_{\vec{\alpha}(1)}, \vec{\beta}(1))$$

in the following way. Let γ, a path over $\vec{\alpha}(0)$ based at $\vec{\beta}(0)$, represent an element in $\pi_1(Z_{\vec{\alpha}(0)}, \vec{\beta}(0))$. Moreover, let $H : I \times I \to Z_0$ be a homotopy with the property that $H(\cdot, 0) = \gamma$ and $H(\cdot, s)$ is a path over $\vec{\alpha}(s)$ based at $\vec{\beta}(s)$ for all $s \in I$. Note that such a homotopy always exists. Then define:

$$\vec{\eta}_* ([\gamma]) := [H(\cdot, 1)].$$

In the following proposition we state that this $[H(\cdot, 1)]$ is well-defined and does not depend on the special choice of H.

Proposition 5.5 *Let $\vec{\eta}_i = \vec{\alpha}_i \times \vec{\beta}_i : I \to (T_0\Delta)^* \times (T_{p_0}D_0)^*$ for $i = 1, 2$ be two paths which are homotopic modulo endpoints. Let moreover $H_i : I \times I \to Z_0$ for $i = 1, 2$ be two homotopies such that $H_i(\cdot, s)$ is a path over $\vec{\alpha}_i(s)$ based at $\vec{\beta}_i(s)$ for any $s \in I$. If there is a homotopy over $\vec{\alpha}_1(0) = \vec{\alpha}_2(0)$ based at $\vec{\beta}_1(0) = \vec{\beta}_2(0)$ between $H_1(\cdot, 0)$ and $H_2(\cdot, 0)$, then there is a homotopy over $\vec{\alpha}_1(1) = \vec{\alpha}_2(1)$ based at $\vec{\beta}_1(1) = \vec{\beta}_2(1)$ between $H_1(\cdot, 1)$ and $H_2(\cdot, 1)$.*

Proof: We know that there is a homotopy, say $\mathcal{H} : I \times I \to Z_0$, between $H_1(\cdot, 1)$ and $H_2(\cdot, 1)$ by using first H_1 as homotopy between $H_1(\cdot, 1)$ and $H_1(\cdot, 0)$, then using the homotopy between $H_1(\cdot, 0)$ and $H_2(\cdot, 0)$ and finally using H_2 as homotopy between $H_2(\cdot, 0)$ and $H_2(\cdot, 1)$. The question is: Can such a homotopy be realized as homotopy over $\vec{\alpha}_1(1) = \vec{\alpha}_2(1)$ based at $\vec{\beta}_1(1) = \vec{\beta}_2(1)$? Define $\vec{\lambda} := \vec{\alpha}_1^{-1} \star \vec{\alpha}_i(0) \star \vec{\alpha}_2$ and $\vec{\mu} := \vec{\beta}_1^{-1} \star \vec{\beta}_i(0) \star \vec{\beta}_2$. Then we can formulate the question more precisely: Given closed paths $\vec{\lambda} : I \to (T_0\Delta)^*$ and $\vec{\mu} : I \to (T_{p_0}D_0)^*$, which are homotopic modulo endpoints to the respective constant paths and given a homotopy $\mathcal{H} : I \times I \to Z_0$ such that $\mathcal{H}(\cdot, s)$ is a path over $\vec{\lambda}(s)$ based at $\vec{\mu}(s)$, can we then find a homotopy $\tilde{\mathcal{H}} : I \times I \to Z_0$ between $\mathcal{H}(\cdot, 0)$ and $\mathcal{H}(\cdot, 1)$ such that $\mathcal{H}(\cdot, s)$ is a path over $\vec{\lambda}(0) = \vec{\lambda}(1)$ based at $\vec{\mu}(0) = \vec{\mu}(1)$ for all $s \in I$?

We give a positive answer to this question by doing the following calculation locally around each double point of Z_0.

Let t be a coordinate on Δ such that $\frac{\partial}{\partial t} = \vec{\lambda}(0) = \vec{\lambda}(1)$. Write $\vec{\lambda}(\tau) = l(\tau)\,\frac{\partial}{\partial t}$. Choose around each double point p_{kl} local coordinates (x, y) in Z such that the function h in the coordinates (x, y) and t looks like:

$$h(x, y) = x \cdot y.$$

For the double point p_0 we may assume that y is the coordinate on D_0 and that $\frac{\partial}{\partial y} = \vec{\mu}(0) = \vec{\mu}(1)$. Write $\vec{\mu}(\tau) = m(\tau)\,\frac{\partial}{\partial y}$.

Let $\vec{L} : I \times I \to (T_0\Delta)^*$ resp. $\vec{M} : I \times I \to (T_{p_0}D_0)^*$ be the homotopies modulo endpoints between $\vec{\lambda}$ resp. $\vec{\mu}$ and the constant path $\vec{\lambda}(0) = \vec{\lambda}(1)$ resp. $\vec{\mu}(0) = \vec{\mu}(1)$. Write $\vec{L}(\tau, s) = L(\tau, s)\,\frac{\partial}{\partial t}$ and $\vec{M}(\tau, s) = M(\tau, s)\,\frac{\partial}{\partial y}$.

By $\sigma : \mathbb{R}^{\geq 0} \to [0; 1]$ we denote a bump function which is $\equiv 1$ near 0 and $\equiv 0$ on $\mathbb{R}^{\geq r_0}$ for a sufficiently small r_0 and strictly decreasing inbetween.

For $(\tau, s) \in I \times I$ where $\mathcal{H}(\tau, s)$ lies in an x-coordinate, we describe $x \circ \mathcal{H}$ by polar coordinates:

$$x \circ \mathcal{H}(\tau, s) = r(\tau, s)\, e^{2\pi i \varphi(\tau, s)}.$$

Then modify the homotopy $\mathcal{H}$ to a homotopy $\tilde{\mathcal{H}}$ by setting:

$$x \circ \tilde{\mathcal{H}}(\tau, s) := \frac{1}{l(\sigma(r(\tau, s)) \cdot s)}\, x \circ \mathcal{H}(\tau, s).$$

Around the point p_0 we define $\tilde{\mathcal{H}}$ by:

$$x \circ \tilde{\mathcal{H}}(\tau, s) := \frac{m(\sigma(r(\tau, s)) \cdot s)}{l(\sigma(r(\tau, s)) \cdot s)}\, x \circ \mathcal{H}(\tau, s).$$

Note that $\tilde{\mathcal{H}}$ is a homotopy over $\vec{\lambda}(0) = \vec{\lambda}(1)$ based at $\vec{\mu}(0) = \vec{\mu}(1)$.

It remains to show that there is a homotopy over $\vec{\lambda}(0) = \vec{\lambda}(1)$ based at $\vec{\mu}(0) = \vec{\mu}(1)$ between $\tilde{\mathcal{H}}(\cdot, 1)$ and $\mathcal{H}(\cdot, 1)$.

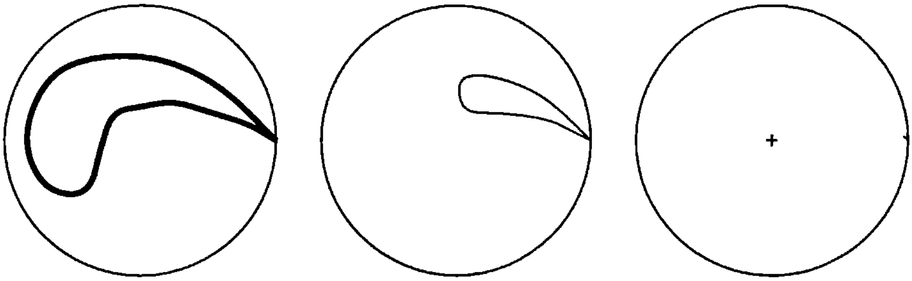


Figure 5.2: Illustration of a homotopy between a path $\vec{\lambda}$ in $T_0\Delta$ and the constant path $\vec{\lambda}(0) = \vec{\lambda}(1)$.

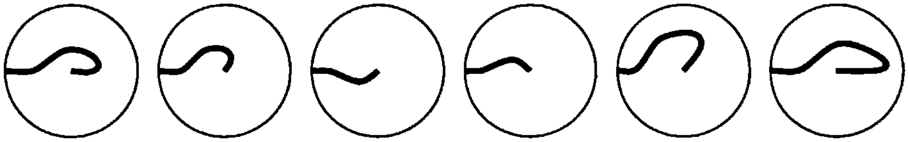


Figure 5.3: Illustration of a homotopy $\mathcal{H}$ (over the path $\vec{\lambda}$) at a double point (other than p_0) in the x-coordinate.

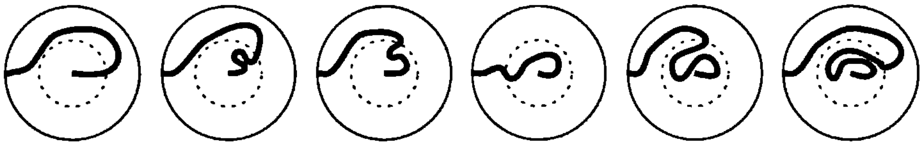


Figure 5.4: Illustration of the homotopy $\tilde{\tilde{\mathcal{H}}}$ at a double point (other than p_0) in the x-coordinate.

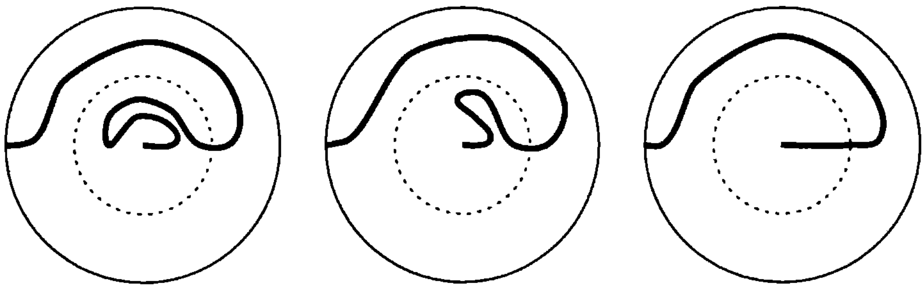


Figure 5.5: Illustration of the homotopy between $\mathcal{H}(\cdot, 1)$ and $\tilde{\tilde{\mathcal{H}}}(\cdot, 1)$ at a double point (other than p_0) in the x-coordinate.

This homotopy is realized by the the following deformation of $\mathcal{H}(\cdot,1)$ in the neighbourhood of each double point:

$$(\tau,s) \mapsto \frac{1}{L(\sigma(r(\tau,1)),s)}\, x \circ \mathcal{H}(\tau,1)$$

respectively around p_0:

$$(\tau,s) \mapsto \frac{M(\sigma(r(\tau,1)),s)}{L(\sigma(r(\tau,1)),s)}\, x \circ \mathcal{H}(\tau,1).$$

□

5.5 The Local System of Fundamental Groups on the Fibration

The locally trivial fibration $h : Z^* \to \Delta^*$ together with a section $\sigma : \Delta \to Z$ defines a local system on Δ^* by associating $\pi_1(Z_t, \sigma(t))$ to a point $t \in \Delta^*$. For a homotopy class modulo endpoints represented by a path in Δ^* one considers trivializations of the fibration over this path, which have σ as a constant section. Such a homotopy induces a map from the fundamental group over the starting point to the fundamental group over the endpoint.

However, here we would like to consider a local system on $\Delta^* \times D_0^*$. The idea is that such a section σ corresponds to a point p in Z_0 by taking $p = \sigma(0)$. We want to consider those sections σ, where $\sigma(0) \in D_0$. In order to be able to define morphisms we perform the following construction.

Let t resp. p be coordinates on Δ resp. D_0 such that Δ resp. D_0 are open disks with radius 1 in these coordinates. Consider the map

$$\begin{aligned} \pi : \Delta \times D_0^* &\longrightarrow \Delta \\ (t,p) &\longmapsto |p| \cdot t. \end{aligned}$$

Observe that all fibers of this map are punctured disks but that π is not a locally trivial fibration. But nonetheless, if we define for any $\delta > 0$ the annulus $D_0^\delta := \{p \in D_0 \big| |p| > \delta\}$, then the restriction of π to $\Delta \times D_0^\delta$ is a locally trivial fibration.

Let now $P : \Delta \times D_0^* \to Z$ be a continuous map such that the following diagram commutes:

$$\begin{tikzcd} \Delta \times D_0^* \arrow[rr, "P"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "h"] \\ & \Delta & \end{tikzcd}$$

and such that $P_{|\{0\}\times D_0^*} : D_0^* \to D_0^*$ is the identity.

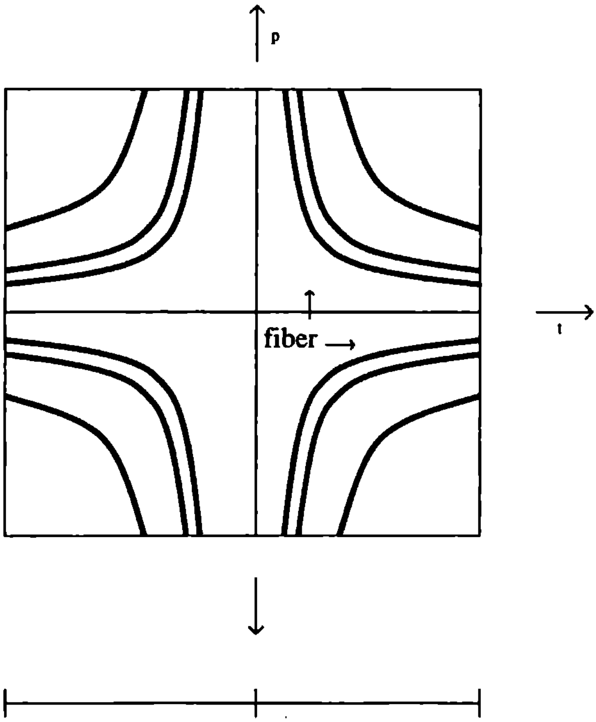


Figure 5.6: Picture of the map π (restricted to the real coordinates in $\Delta \times D_0^*$ and Δ).

For $\delta > 0$ we define $\Delta_\delta := \{t \in \Delta \big| |t| < \delta\}$ and $\Delta_\delta^* := \Delta_\delta \setminus \{0\}$. Then we have for any $p \in D_0^\delta$ a section of $\pi_{|\Delta \times D_0^\delta}$

$$\begin{aligned} \tilde{\sigma}_p : \Delta_\delta &\longrightarrow \Delta \times D_0^\delta \\ \tau &\longmapsto \left(\tfrac{\tau}{|p|}, p\right). \end{aligned}$$

Composed with P we obtain a section of $h_{|h^{-1}(\Delta_\delta)}$, which we denote by σ_p : $\Delta_\delta \to Z$. Note that holds

$$\sigma_p(0) = p.$$

In the following, we will for any $0 < \delta < 1$ define a local system of groups on $\Delta_\delta^* \times D_0^\delta$. This system will be called *the local system of fundamental groups on the fibration* and we denote it by:

$$\boxed{\{\pi_1(Z_t,\, \sigma_p(t))\}_{(t,p)\in\Delta_\delta^* \times D_0^\delta}}.$$

To a point (t,p) in $\Delta_\delta^* \times D_0^\delta$ we associate the group

$$\pi_1\left(Z_t, \sigma_p(t)\right).$$

With a homotopy class modulo endpoints represented by a path

$$\eta = \alpha \times \beta : I \longrightarrow \Delta_\delta^* \times D_0^\delta$$

we associate a group isomorphism

$$\eta_* : \pi_1 \left(Z_{\alpha(0)}, \sigma_{\beta(0)}(\alpha(0)) \right) \longrightarrow \pi_1 \left(Z_{\alpha(1)}, \sigma_{\beta(1)}(\alpha(1)) \right).$$

in the following way.

Let $\gamma : I \to Z_{\alpha(0)}$ represent an element of $\pi_1 \left(Z_{\alpha(0)}, \sigma_{\beta(0)}(\alpha(0)) \right)$ and let $H : I \times I \to Z$ be a continuous map such that $H(\cdot, 0) = \gamma$ and $H(\cdot, s)$ is a path in $Z_{\alpha(s)}$ based at $\sigma_{\beta(s)}(\alpha(s))$. Then define:

$$\eta_* ([\gamma]) := [H(\cdot, 1)] \in \pi_1 \left(Z_{\alpha(1)}, \sigma_{\beta(1)}(\alpha(1)) \right).$$

We will see that this is well-defined in the following way. Define

$$\mathring{\Delta}_\delta := \left\{ \left(\rho, e^{2\pi i \theta} \right) \in \mathbb{R}^{\geq 0} \times S^1 \middle| \rho \cdot e^{2\pi i \theta} \in \Delta_\delta \right\}.$$

In the next section we are going to define a local system on $\mathring{\Delta}_\delta \times D_0^\delta$ such that the local system above is its restriction to $\Delta_\delta^* \times D_0^\delta$ by means of the obvious embedding

$$\Delta_\delta^* \times D_0^\delta \hookrightarrow \mathring{\Delta}_\delta \times D_0^\delta.$$

5.6 Comparison of the two Local Systems

In this section we will compare the local system of nearby fundamental groups with the local system of fundamental groups on the fibration. The link between these two will be provided by a local system, which we call *the local system of fundamental groups on the real blow-up of the fibration* and which we are going to construct in the sequel.

5.6.1 The Real Blow-up of the Fibration

We want to define an extension of Z^* to a topological space $\mathring{Z}$ and a map

$$\mathring{h} : \mathring{Z} \longrightarrow \mathring{\Delta},$$

in such a way that the diagram

$$\begin{array}{ccc} \mathring{Z} & \hookleftarrow & Z^* \\ \mathring{h} \downarrow & & \downarrow h^* \\ \mathring{\Delta} & \hookleftarrow & \Delta^* \end{array}$$

commutes and that $\mathring{h} : \mathring{Z} \to \mathring{\Delta}$ is a locally trivial topological fibration.

Choose a coordinate t on Δ and choose local coordinates $(x, y) : W_{kl} = U^k_{kl} \times U^l_{kl} \to \mathbb{C}^2$ around any double point $p_{kl} \in Z$ for $[k < l]$ such that the function h in these coordinates looks like

$$h(x,y) = x \cdot y \qquad \text{for all } |x| < 2,\ |y| < 2.$$

Consider the map

$$\begin{array}{rcl} \chi : \left(\mathbb{R}^{\geq 0} \times S^1\right) \times \left(\mathbb{R}^{\geq 0} \times S^1\right) & \longrightarrow & \left(\mathbb{R}^{\geq 0} \times S^1\right) \\ \left((R_1, e^{2\pi i \phi_1}) \ , \ (R_2, e^{2\pi i \phi_2})\right) & \longmapsto & \left(R_1 R_2, e^{2\pi i (\phi_1 + \phi_2)}\right). \end{array}$$

Then define for any pair k, l with $[k < l]$:

$$\overset{\circ}{U}{}^k_{kl} := \left\{ \left(R, e^{2\pi i \phi}\right) \mathbb{R}^{\geq 0} \times S^1 \middle| Re^{2\pi i \phi} \in x(U^k_{kl}) \right\}$$

and

$$\overset{\circ}{U}{}^l_{kl} := \left\{ \left(R, e^{2\pi i \phi}\right) \mathbb{R}^{\geq 0} \times S^1 \middle| Re^{2\pi i \phi} \in y(U^l_{kl}) \right\}.$$

There are the obvious maps $\overset{\circ}{U}{}^k_{kl} \to U^k_{kl}$ and $\overset{\circ}{U}{}^l_{kl} \to U^l_{kl}$.

We construct $\overset{\circ}{Z}$ as follows. Recall that there are natural maps $\overset{\circ}{\Delta} \to \Delta$ and $\Delta^* \hookrightarrow \overset{\circ}{\Delta}$. Define first:

$$\breve{Z} := Z \times_{\Delta} \overset{\circ}{\Delta}.$$

In $\breve{Z}$ we have the subspaces

$$\begin{array}{rcl} \breve{W}_{kl} & := & W_{kl} \times_{\Delta} \overset{\circ}{\Delta} = \left(U^k_{kl} \times U^l_{kl}\right) \times_{\Delta} \overset{\circ}{\Delta} \\ \breve{W}^*_{kl} & := & \left(U^{*k}_{kl} \times U^{*l}_{kl}\right) \times_{\Delta^*} \Delta^*. \end{array}$$

Now we use $\chi_{|\overset{\circ}{U}{}^k_{kl} \times \overset{\circ}{U}{}^l_{kl}}$ and the identity map to define

$$\overset{\circ}{W}_{kl} := \left(\overset{\circ}{U}{}^k_{kl} \times \overset{\circ}{U}{}^l_{kl}\right) \times_{\overset{\circ}{\Delta}} \overset{\circ}{\Delta}.$$

Finally we construct $\overset{\circ}{Z}$ by replacing $\breve{W}_{kl}$ in $\breve{Z}$ by $\overset{\circ}{W}_{kl}$, glued along the common subspace W^*_{kl}.

The projection to the second factor in the above mentioned fibered products yields the well-defined map we are looking after:

$$\overset{\circ}{h} : \overset{\circ}{Z} \longrightarrow \overset{\circ}{\Delta}.$$

Note that the maps $\overset{\circ}{U}{}^k_{kl} \to U^k_{kl}$ and $\overset{\circ}{U}{}^l_{kl} \to U^l_{kl}$ induce a map $\overset{\circ}{Z} \to Z$ over $\overset{\circ}{\Delta} \to \Delta$.

Remark: There is a coordinate-free way to construct $\overset{\circ}{Z}$ by using the theory of log-structures (cf. e. g. [KN95], [Ill94], [Kat89], [Ste95]). We decided not to introduce the language of log-structures here since we use the space $\overset{\circ}{Z}$ just as a link between the two earlier defined local systems and because we need the coordinates anyway. The following theorem is proved in greater generality by Usui [Usu96].

Theorem 5.6 *$\overset{\circ}{h}:\overset{\circ}{Z}\to\overset{\circ}{\Delta}$ is a locally trivial topological fibration.*

Proof: We are given – as before – local coordinates $(x,y): W_{kl} = U^k_{kl} \times U^l_{kl} \to \mathbb{C}^2$ around any double point p_{kl} for $[k<l]$ and t on Δ such that $h(x,y) = xy$. We identify W_{kl} and $U^k_{kl} \times U^l_{kl}$ by these coordinates.

Define the polycylinders

$$P_{kl} := \{(x,y)\,\big|\,|x|<1,\ |y|<1\} \subset W_{kl}$$

and for κ with $0<\kappa<\frac{1}{2}$

$$P^{\kappa}_{kl} := \{(x,y)\,\big|\,|x|<\kappa,\ |y|<\kappa\} \subset W_{kl}.$$

Let

$$\check{Z} := Z \setminus \bigcup_{[k<l]} \bar{P}_{kl} \text{ and } \check{Z}^{\kappa} := Z \setminus \bigcup_{[k<l]} \bar{P}^{\kappa}_{kl}.$$

Then $h_{|\check{Z}} : \left(\check{Z}, \bigcup\limits_{[k<l]} \partial\bar{P}_{kl}\right) \to \Delta$ is a trivial fibration by the Ehresmann fibration theorem. Let $\check{Z}_0 := h^{-1}(0) \cap \check{Z}$ and observe that $\partial\bar{P}_{kl} \cap \check{Z}_0$ is homeomorphic with $S^1 \coprod S^1$. Then there exists a trivialization

$$\begin{tikzcd}
\left(\check{Z}, \bigcup\limits_{[k<l]} \partial\bar{P}_{kl}\right) \arrow[r, "\Phi"] \arrow[dr, "h"'] & \left(\check{Z}_0, \bigcup\limits_{[k<l]} (S^1 \coprod S^1)\right) \times \Delta \arrow[d, "pr_2"] \\
& \Delta
\end{tikzcd}$$

Let $C_{kl} := S^1 \times ([0,\kappa] \cup [1-\kappa, 1])$ be several copies of the cylinder $S^1 \times I$ deprived of an annulus in the middle, which are in one-to-one correspondence with the double points p_{kl} for $[k<l]$.

Assume for the moment that we have for each pair k,l (with $[k<l]$) a trivialization

$$\begin{tikzcd}
(\bar{P}_{kl} \setminus P^{\kappa}_{kl}, \partial\bar{P}_{kl}) \arrow[r, "\psi_{kl}"] \arrow[dr, "h"'] & (C_{kl}, \partial C_{kl}) \times \Delta \arrow[d, "pr_2"] \\
& \Delta
\end{tikzcd},$$

which we put together to one trivialization Ψ of $\bigcup\limits_{[k<l]} (\bar{P}_{kl} \setminus P^{\kappa}_{kl}, \partial\bar{P}_{kl})$.

Given such a trivialization, we claim that we can extend Φ to a trivialization Φ^κ of $\check{Z}^\kappa$ such that Φ^κ and Ψ coincide on $\bigcup_{[k<l]} \partial\bar{P}^\kappa_{kl}$.

This claim boils down to the following well-known assertion: Given an orientation preserving homeomorphism

$$\begin{aligned} a : \{1\} \times S^1 &\longrightarrow \quad \{1\} \times S^1 \\ (1, e^{2\pi\imath\tau}) &\longmapsto \left(1, e^{2\pi\imath\alpha(\tau)}\right), \end{aligned}$$

then there is a homeomorphism $A : I \times S^1 \to I \times S^1$ such that $A_{|\{1\}\times S^1} = a$ and $A_{|\{0\}\times S^1} = id$. This A is given by : $A\left(t, e^{2\pi\imath\tau}\right) = \left(t, e^{2\pi\imath(t\,\tau+(1-t)\alpha(\tau))}\right)$. Note that for $\alpha : \mathbb{R} \to \mathbb{R}$ holds $\alpha(\tau+n) = \alpha(\tau)+n$ for $n \in \mathbb{Z}$ and that therefore A is well-defined.

It is clear that all these trivializations over Δ may be pulled back to trivializations over $\mathring{\Delta}$.

On the other hand define:

$$\mathring{P}_{kl} := \left\{\left(\left(R_1, e^{2\pi\imath\phi_1}\right), \left(R_2, e^{2\pi\imath\phi_2}\right)\right) \in \left(\mathbb{R}^{\geq 0} \times S^1\right) \times \left(\mathbb{R}^{\geq 0} \times S^1\right) \,\middle|\, {R_1<1 \atop R_2<1}\right\}$$

and

$$\mathring{P}^\kappa_{kl} := \left\{\left(\left(R_1, e^{2\pi\imath\phi_1}\right), \left(R_2, e^{2\pi\imath\phi_2}\right)\right) \in \left(\mathbb{R}^{\geq 0} \times S^1\right) \times \left(\mathbb{R}^{\geq 0} \times S^1\right) \,\middle|\, {R_1<\kappa \atop R_2<\kappa}\right\}.$$

In $\mathring{P}_{kl}$ we take as typical fiber of $\mathring{h}_{|\mathring{P}_{kl}}$ the fiber over $(0, e^{2\pi\imath 0}) = (0, 1)$:

$$\begin{aligned} F_{(0,1)} &:= \left\{\left(\left(r_1, e^{2\pi\imath\varphi_1}\right), \left(r_2, e^{2\pi\imath\varphi_2}\right)\right) \in \mathring{P}_{kl} \,\middle|\, r_1 \cdot r_2 = 0,\, \varphi_1 + \varphi_2 \equiv 0\right\} \\ &= \left\{\left(\left(r_1, e^{2\pi\imath\varphi_1}\right), \left(0, e^{2\pi\imath\varphi_2}\right)\right) \in \mathring{P}_{kl} \,\middle|\, \varphi_1 \equiv -\varphi_2 \bmod \mathbb{Z}\right\} \\ &\cup \left\{\left(\left(0, e^{2\pi\imath\varphi_1}\right), \left(r_2, e^{2\pi\imath\varphi_2}\right)\right) \in \mathring{P}_{kl} \,\middle|\, \varphi_2 \equiv -\varphi_1 \bmod \mathbb{Z}\right\}. \end{aligned}$$

Then consider for $\theta \in \mathbb{R}$ the family of self-homeomorphisms $F_{(0,1)}$:

$$g_\theta : \begin{array}{l} \left(\left(r_1, e^{2\pi\imath\varphi_1}\right), \left(0, e^{-2\pi\imath\varphi_1}\right)\right) \mapsto \left(\left(r_1, e^{2\pi\imath(\varphi_1+r_1\frac{\theta}{2})}\right), \left(0, e^{-2\pi\imath(\varphi_1+r_1\frac{\theta}{2})}\right)\right) \\ \left(\left(0, e^{-2\pi\imath\varphi_2}\right), \left(r_2, e^{2\pi\imath\varphi_2}\right)\right) \mapsto \left(\left(0, e^{-2\pi\imath(\varphi_2+r_2\frac{\theta}{2})}\right), \left(r_2, e^{2\pi\imath(\varphi_2+r_2\frac{\theta}{2})}\right)\right) \end{array}$$

For an open set $\mathring{V}$ in $\mathring{\Delta}$ of the form $\mathring{V} = [0; r_0[\times \left\{e^{2\pi\imath\theta} \middle| \alpha_0 < \theta < \beta_0\right\}$ and $0 < \beta_0 - \alpha_0 < 1$ we define the trivialization:

$$\Theta_{\mathring{V}} : \mathring{h}^{-1}(\mathring{V}) \cap \mathring{P}_{kl} \longrightarrow \mathring{V} \times F_{(0,1)},$$

which sends an element $\left(\left(R_1, e^{2\pi\imath\phi_1}\right), \left(R_2, e^{2\pi\imath\phi_2}\right)\right)$, where $\phi_1 + \phi_2 \in]\alpha_0; \beta_0[$ to

$$\left(\left(R_1 R_2, e^{2\pi\imath(\phi_1+\phi_2)}\right), g_{\phi_1+\phi_2}\left(\frac{R_1 - R_2}{1 - R_1 R_2}, e^{2\pi\imath\left(\phi_1 - \frac{\phi_1+\phi_2}{2}\right)}, 0, e^{2\pi\imath\left(\phi_2 - \frac{\phi_1+\phi_2}{2}\right)}\right)\right)$$

if $R_1 \geq R_2$ and to

$$\left((R_1 R_2, e^{2\pi i(\phi_1+\phi_2)}), g_{\phi_1+\phi_2}\left(0, e^{2\pi i\left(\phi_1 - \frac{\phi_1+\phi_2}{2}\right)}; \frac{R_2-R_1}{1-R_1R_2}, e^{2\pi i\left(\phi_2-\frac{\phi_1+\phi_2}{2}\right)}\right)\right)$$

if $R_2 \geq R_1$. It is a straightforward calculation to check that this map is indeed a homeomorphism. Note that $\frac{\phi_1+\phi_2}{2}$ is well-defined since $\alpha_0 < \phi_1 + \phi_2 < \beta_0$ and $0 < \beta_0 - \alpha_0 < 1$.

Moreover, observe that on $\partial \overset{\circ}{\bar{P}}_{kl}$ holds:

$$\Theta_{\overset{\circ}{V}}\left((1, e^{2\pi i\phi_1}), (R_2, e^{2\pi i\phi_2})\right) = \left(\left(R_2, e^{2\pi i(\phi_1+\phi_2)}\right), \left(1, e^{2\pi i\phi_1}; 0, e^{2\pi i\phi_2}\right)\right)$$

$$\Theta_{\overset{\circ}{V}}\left((R_1, e^{2\pi i\phi_1}), (1, e^{2\pi i\phi_2})\right) = \left(\left(R_1, e^{2\pi i(\phi_1+\phi_2)}\right), \left(0, e^{2\pi i\phi_1}; 1, e^{2\pi i\phi_2}\right)\right).$$

This indicates that

$$\Theta_{\overset{\circ}{V}}\Big|_{\bigcup_{[k<l]}\left(\overset{\circ}{\bar{P}}_{kl}\setminus \overset{\circ}{P}{}^{\kappa}_{kl}\right)\cap \overset{\circ}{h}{}^{-1}(\overset{\circ}{V})}$$

can be extended to a global trivialization Θ of $\bigcup_{[k<l]} \left(\bar{P}_{kl} \setminus P^{\kappa}_{kl}\right)$ over Δ respectively of $\bigcup_{[k<l]} \left(\overset{\circ}{\bar{P}}_{kl} \setminus \overset{\circ}{P}{}^{\kappa}_{kl}\right)$ over $\overset{\circ}{\Delta}$. Hence we may extend the trivialization Φ of $\check{Z}$ to a trivialization Φ^{κ} of $\check{Z}^{\kappa}$, which coincides with Θ on $\bigcup_{[k<l]} \partial \overset{\circ}{\bar{P}}_{kl}$.

This shows that we obtain for any open $\overset{\circ}{V}$ in $\overset{\circ}{\Delta}$ like above a trivialization of $\overset{\circ}{h}{}^{-1}(\overset{\circ}{V})$ over $\overset{\circ}{V}$, which is to say that $\overset{\circ}{h}$ is locally trivial. □

Having this theorem at hand we can define a third local system of groups, this time on $\overset{\circ}{\Delta}_\delta \times D_0^\delta$. The purpose of this local system here is that it serves as link between the local system of nearby fundamental groups and the local system of fundamental groups on the fibration. This third local system will be referred to as *the local system of fundamental groups on the real blow-up of the fibration* or briefly *the local system of the real blow-up.* We denote it by:

$$\boxed{\left\{\pi_1\left(\overset{\circ}{Z}_{(\rho, e^{2\pi i\theta})}, \overset{\circ}{\sigma}_p\left(\rho, e^{2\pi i\theta}\right)\right)\right\}_{\left((\rho, e^{2\pi i\theta}), p\right)\in \overset{\circ}{\Delta}_\delta \times D_0^\delta}}.$$

It is an extension of the local system of fundamental groups on the fibration on $\Delta_\delta^* \times D_0^\delta$ to $\overset{\circ}{\Delta}_\delta \times D_0^\delta$ and it is defined as follows:

Observe that for $p \in D_0^\delta$ the section $\sigma_p : \Delta_\delta \to Z$ can be pulled back in a unique way to a section $\overset{\circ}{\sigma}_p : \overset{\circ}{\Delta}_\delta \to \overset{\circ}{Z}$.

To any $((\rho, e^{2\pi i\theta}), p) \in \overset{\circ}{\Delta}_\delta \times D_0^\delta$ we associate the group

$$\pi_1\left(\overset{\circ}{Z}_{(\rho, e^{2\pi i\theta})}, \overset{\circ}{\sigma}_p\left(\rho, e^{2\pi i\theta}\right)\right)$$

and to a homotopy class modulo endpoints represented by a path

$$\eta = \alpha \times \beta : I \longrightarrow \overset{\circ}{\Delta}_\delta \times D_0^\delta$$

we associate similarly as before a group isomorphism

$$\eta_* : \pi_1\left(\overset{\circ}{Z}_{\alpha(0)}, \overset{\circ}{\sigma}_{\beta(0)}\left(\alpha(0)\right)\right) \longrightarrow \pi_1\left(\overset{\circ}{Z}_{\alpha(1)}, \overset{\circ}{\sigma}_{\beta(1)}\left(\alpha(1)\right)\right).$$

Let $\gamma : I \longrightarrow \overset{\circ}{Z}$ be an element of $\pi_1\left(\overset{\circ}{Z}_{\alpha(0)}, \overset{\circ}{\sigma}_{\beta(0)}\left(\alpha(0)\right)\right)$ and let $H : I \times I \longrightarrow \overset{\circ}{H}$ be a continuous map such that $H(\cdot, 0) = \gamma$ and for any $s \in I$ is $H(\cdot, s)$ a path in $\overset{\circ}{Z}_{\alpha(s)}$ based at $\overset{\circ}{\sigma}_{\beta(s)}\left(\alpha(s)\right)$.

Then define:

$$\eta_*\left([\gamma]\right) := [H(\cdot, 1)] \in \pi_1\left(\overset{\circ}{Z}_{\alpha(1)}, \overset{\circ}{\sigma}_{\beta(1)}\left(\alpha(1)\right)\right).$$

The following proposition justifies this definition and moreover the definition of the local system of fundamental groups of the fibration.

Proposition 5.7 *Let $\eta_i = \alpha_i \times \beta_i : I \to \overset{\circ}{\Delta}_\delta \times D_0^\delta$ for $i = 1, 2$ be two paths, which are homotopic modulo endpoints. Let moreover $H_i : I \times I \to \overset{\circ}{Z}$ for $i = 1, 2$ be two homotopies such that $H_i(\cdot, s)$ is a path in $\overset{\circ}{Z}_{\alpha_i(s)}$ based at $\overset{\circ}{\sigma}_{\beta_i(s)}\left(\alpha_i(s)\right)$ for any $s \in I$.*

If $H_1(\cdot, 0)$ and $H_2(\cdot, 0)$ represent the same homotopy class in

$$\pi_1\left(\overset{\circ}{Z}_{\alpha_i(0)}, \overset{\circ}{\sigma}_{\beta_i(0)}\left(\alpha_i(0)\right)\right)$$

with $i = 1$ or 2, then $H_1(\cdot, 1)$ and $H_2(\cdot, 1)$ represent the same homotopy class in

$$\pi_1\left(\overset{\circ}{Z}_{\alpha_i(1)}, \overset{\circ}{\sigma}_{\beta_i(1)}\left(\alpha_i(1)\right)\right)$$

with $i = 1$ or 2.

Proof: Similar as in the proof of Proposition 5.5 the assertion may be reduced to: If $\lambda : I \to \overset{\circ}{\Delta}_\delta$ and $\mu : I \to D_0^\delta$ are closed paths, which are homotopic modulo endpoint to the respective constant paths and given a homotopy $\mathcal{H} : I \times I \to \overset{\circ}{Z}$ such that $\mathcal{H}(\cdot, s)$ is a path in $\overset{\circ}{Z}_{\lambda(s)}$ based at $\overset{\circ}{\sigma}_{\mu(s)}\left(\lambda(s)\right)$ for any $s \in I$, then there is a $\tilde{\mathcal{H}} : I \times I \to \overset{\circ}{Z}$, a homotopy between $\mathcal{H}(\cdot, 0)$ and $\mathcal{H}(\cdot, 1)$ such that $\tilde{\mathcal{H}}(\cdot, s)$ is a path in $\overset{\circ}{Z}_{\lambda(0)} = \overset{\circ}{Z}_{\lambda(1)}$ based at $\overset{\circ}{\sigma}_{\mu(0)}(\lambda(0)) = \overset{\circ}{\sigma}_{\mu(1)}(\lambda(1))$ for any $s \in I$.

Observe first that there is a homotopy $\tilde{\tilde{\mathcal{H}}} : I \times I \to \overset{\circ}{Z}_{\lambda(0)} = \overset{\circ}{Z}_{\lambda(1)}$ between $\mathcal{H}(\cdot, 0)$ and $\mathcal{H}(\cdot, 1)$ lying completely in the one fiber over $\lambda(0) = \lambda(1)$ but not necessarily leaving the base point $\overset{\circ}{\sigma}_{\mu(0)}(\lambda(0)) = \overset{\circ}{\sigma}_{\mu(1)}(\lambda(1))$ fixed. This is true for the following reason.

Let $L : I \times I \to \overset{\circ}{\Delta}_\delta$ be the homotopy between λ and the constant path $\lambda(0) = \lambda(1)$. Then the pull-back bundle $L^* \overset{\circ}{Z}$ is globally trivial, i. e. there is a trivialization

$$\begin{tikzcd} L^* \overset{\circ}{Z} \arrow[rr, "\Phi"] \arrow[dr, "L^* h"'] & & (I \times I) \times \overset{\circ}{Z}_{\lambda(0)} \arrow[dl, "pr_2"] \\ & I \times I & \end{tikzcd}$$

The homotopy $\mathcal{H}$ can be considered as a map to $L^* \overset{\circ}{Z}$ (since $\mathcal{H}(\cdot, s)$ is a path in $\overset{\circ}{Z}_{\lambda(s)}$ for $s \in I$) and therefore to $(I \times I) \times \overset{\circ}{Z}_{\lambda(0)}$. The projection to the second factor is a homotopy in $\overset{\circ}{Z}_{\lambda(0)}$ between $\mathcal{H}(\cdot, 0)$ and $\mathcal{H}(\cdot, 1)$.

If the path, which is formed by the trace of the base point is zero-homotopic, then we may find a homotopy between $\mathcal{H}(\cdot, 0)$ and $\mathcal{H}(\cdot, 1)$ leaving the base point $\overset{\circ}{\sigma}_{\mu(0)}(\lambda(0)) = \overset{\circ}{\sigma}_{\mu(1)}(\lambda(1))$ fixed.

Note that a path in $(I \times I) \times \overset{\circ}{Z}_{\lambda(0)}$ is zero homotopic if and only if its projection to $\overset{\circ}{Z}_{\lambda(0)}$ is.

The above mentioned trace of the base point is the projection to $\overset{\circ}{Z}_{\lambda(0)}$ of the path

$$\phi(\mathcal{H}(0, \cdot)) = \phi(\mathcal{H}(1, \cdot)).$$

But this path is zero-homotopic iff $\mathcal{H}(0, \cdot) = \mathcal{H}(1, \cdot)$ is zero-homotopic. Note that $\mathcal{H}(0, s) = \mathcal{H}(1, s) = \overset{\circ}{\sigma}_{\mu(s)}(\lambda(s))$. Therefore, a homotopy to the constant path $\overset{\circ}{\sigma}_{\mu(0)}(\lambda(0)) = \overset{\circ}{\sigma}_{\mu(1)}(\lambda(1))$ is given by:

$$(s, \tau) \longmapsto \overset{\circ}{\sigma}_{M(s,\tau)}(L(s, \tau)).$$

$\square$

Now, we can formulate the main theorem of this section. $(T_0\Delta)^*$, $(T_{p_0}D_0)^*$, Δ_δ^* and D_0^δ are oriented by their complex structures.

Theorem 5.8 *Any pair of orientation preserving homotopy equivalences (for a $\delta > 0$)*

$$(T_0\Delta)^* \longrightarrow \Delta_\delta^* \quad \textit{and} \quad (T_{p_0}D_0)^* \longrightarrow D_0^\delta$$

induces an isomorphism of the local system of nearby fundamental groups on $(T_0\Delta)^* \times (T_{p_0}D_0)^*$ with the local system on $\Delta_\delta^* \times D_0^\delta$ of fundamental groups on the fibration.

Proof: Observe that $\Delta_\delta^* \subset \mathring{\Delta}_\delta$ and call $S := \mathring{\Delta}_\delta \setminus \Delta_\delta^*$. By definition of both local systems, the inclusion $\Delta_\delta^* \times D_0^\delta \subset \mathring{\Delta}_\delta \times D_0^\delta$ induces an isomorphism of the local system of fundamental groups on the fibration with the local system of the real blow up. The latter local system is also equivalent to its restriction to $S \times D_0^\delta$ by the inclusion $S \times D_0^\delta \subset \mathring{\Delta}_\delta \times D_0^\delta$. We call this restriction to $S \times D_0^\delta$ *the local system on the soul of the real blow-up.*

Hence, the assertion of the theorem is equivalent with: Any pair of orientation preserving homotopy equivalences

$$a : (T_0\Delta)^* \longrightarrow S \quad \text{and} \quad b : (T_{p_0}D_0)^* \longrightarrow D_0^\delta$$

induces an isomorphism of the local system of nearby fundamental groups to the local system on the soul of the real blow-up.

According to our remarks in 5.1 we have to show that there is an isomorphism between the group associated to one pair of tangent vectors

$$(\vec{v}_0, \vec{w}_0) \in (T_0\Delta)^* \times (T_{p_0}D_0)^*$$

and the group associated to

$$(a(\vec{v}_0), b(\vec{w}_0)) \in S \times D_0^\delta,$$

which is equivariant with respect to the action of

$$\pi_1\left((T_0\Delta)^* \times (T_{p_0}D_0)^*, (\vec{v}_0, \vec{w}_0)\right)$$

and the action of

$$\pi_1\left(S \times D_0^\delta, (a(\vec{v}_0), b(\vec{w}_0))\right).$$

Let us first construct the isomorphism and show that it is equivariant. Associated to each double point p_{kl} for $[k < l]$ we have from the definition of $\mathring{Z}$ coordinates $\left(R_1, e^{2\pi i\phi_1}\right)$, $\left(R_2, e^{2\pi i\phi_2}\right)$ in $\mathring{Z}$ and coordinates $\left(\rho, e^{2\pi i\theta}\right)$ on $\mathring{\Delta}$ such that in these coordinates

$$\mathring{h}\left(\left(R_1, e^{2\pi i\phi_1}\right), \left(R_2, e^{2\pi i\phi_2}\right)\right) = \left(R_1R_2, e^{2\pi i(\phi_1+\phi_2)}\right).$$

Here we may assume that $a(\vec{v}_0) = (0,1) \in S$ and $b(\vec{w}_0) = (1,0) \in \mathring{\tilde{D}}_0$.

We will define an isomorphism

$$\mathcal{J} : \pi_1\left(Z_{\vec{v}_0}, \vec{w}_0\right) \longrightarrow \pi_1\left(\mathring{Z}_{(0,1)}, \mathring{\sigma}_{(1,0)}\,(0,1)\right).$$

Let $[\gamma]$ be a nearby homotopy class in $\pi_1\left(Z_{\vec{v}_0}, \vec{w}_0\right)$ represented by the path γ over $\vec{v}_0$ based at $\vec{w}_0$.

Note that any piecewise smooth path in $\mathbb{C}$ has a unique lifting with respect to the map

$$\begin{array}{cccc} \mathring{\mathbb{C}} := & \mathbb{R}^{\geq 0} \times S^1 & \to & \mathbb{C} \\ & (r, e^{2\pi i\varphi}) & \mapsto & r \cdot e^{2\pi i\varphi}. \end{array}$$

Therefore, there is a unique lifting of $\gamma \in Z_0$ to a path $\overset{\bullet}{\gamma}$ in $\overset{\circ}{Z}$. The condition that γ lies over $\vec{v}_0$ implies that $\overset{\bullet}{\gamma}$ is even in $\overset{\circ}{Z}_{(0,1)}$. and that it is connected. Define then $\overset{\circ}{\gamma} := \sigma \star \overset{\bullet}{\gamma} \star \sigma^{-1}$, where σ is the straight path in $\overset{\circ}{D}_0$ defined by $t \mapsto (t, e^{2\pi\imath 0}) = (1-t, 1)$, then we obtain a path in $\overset{\circ}{Z}_{(0,1)}$, which is based at $\overset{\circ}{\sigma}_{(1,0)}\,(0,1) = (1,0) \in \overset{\circ}{\bar{D}}_0$.

The map $\gamma \mapsto \overset{\circ}{\gamma}$ induces a well-defined map from $\pi_1(Z_{\vec{v}_0}, \vec{w}_0)$ to $\pi_1\left(\overset{\circ}{Z}_{(0,1)}, \overset{\circ}{\sigma}_{(1,0)}\,(0,1)\right)$, which is certainly a group homomorphism and which is the map $\mathcal{J}$ we wanted to define. It remains to show that $\mathcal{J}$ is an *iso*morphism.

$\mathcal{J}$ is surjective: Call the sets $\left\{(0, e^{2\pi\imath\varphi}),\, (0, e^{-2\pi\imath\varphi}) \middle| \varphi \in [0;1[\right\} \subset \overset{\circ}{P}_{kl} \cap \overset{\circ}{Z}_{(0,1)}$ for $[k<l]$ *vanishing cycles.* We can give $\overset{\circ}{Z}_{(0,1)}$ a differentiable structure in such a way that it coincides with the already given differentiable structure off the vanishing cycles coming from $Z_0 \setminus \bigcup_{[k<l]}\{p_{kl}\}$. It is well-known that any element in the fundamental group of a smooth manifold (with boundary) can be represented by a smooth path (cf. e. g. [tD91], II, 1.10). Moreover observe that by a local construction any such smooth path is homotopic to a smooth path, which intersects the vanishing cycles transversally in a finite number of parameter values, the first and the last of which are the only intersections with the vanishing cycle corresponding to p_0. Moreover, we may assume that $\overset{\odot}{\gamma}$ can be written as $\overset{\odot}{\gamma} = \sigma \star \overset{\bullet}{\gamma} \star \sigma^{-1}$ for some closed path $\overset{\bullet}{\gamma}$, which lies, apart from the base point, entirely in D^+.

A path $\overset{\odot}{\gamma}$ in $\overset{\circ}{Z}_{(0,1)}$ with these properties, composed with the map $\overset{\circ}{Z}_{(0,1)} \to Z_0$ (collapsing the vanishing cycles) yields a path γ in $D^+ \subset Z_0$. By reparametrizing the path $\overset{\odot}{\gamma}$ appropriately we may assume that it lies over $\vec{v}_0$ and is based at $\vec{w}_0$. And hence we find: $\overset{\circ}{\gamma} = \overset{\odot}{\gamma}$.

$\mathcal{J}$ is injective: Here we have to convince ourselves from the following fact. If we are given two homotopic paths $\overset{\circ}{\gamma}_1$ and $\overset{\circ}{\gamma}_2$ in $\overset{\circ}{Z}_{(0,1)}$ based at $(1,0) \in \overset{\circ}{\bar{D}}_0$, which are images under $\mathcal{J}$ of two paths γ_1 and γ_2 over $\vec{v}_0$ based at $\vec{w}_0$, then these latter paths are homotopic over $\vec{v}_0$ based at $\vec{w}_0$.

Let $H : I \times I \to \overset{\circ}{Z}_{(0,1)}$ be a homotopy between the paths $\overset{\circ}{\gamma}_1$ and $\overset{\circ}{\gamma}_2$ in $\overset{\circ}{Z}_{(0,1)}$ based at $(1,0) \in \overset{\circ}{\bar{D}}_0$. Using again the differentiable structure on $\overset{\circ}{Z}_{(0,1)}$ like above we may assume that H is smooth (cf. e. g. [tD91], II, 1.10) and that for each $s \in I$ the path $H(\cdot, s)$ intersects the the vanishing cycles transversally in a finite number of parameter values, the first and the last of which are the only intersections with the vanishing cycle corresponding to p_0. Moreover, we may assume that $H(\cdot, s)$ can be written as $H(\cdot, s) = \sigma \star \eta_s \star \sigma^{-1}$ for some closed path η_s, which lies, apart from the base point, entirely in D^+.

Composed with $\overset{\circ}{Z}_{(0,1)} \to Z_0$ we obtain a homotopy between γ_1 and γ_2 over $\vec{v}_0$ based at $\vec{w}_0$.

Finally, we study the monodromy actions. Let

$$\vec{\eta} = \vec{\alpha} \times \vec{\beta} : I \longrightarrow (T_0\Delta)^* \times (T_{p_0}D_0)^*$$

be a path representing a morphism in the fundamental groupoid of $(T_0\Delta)^* \times (T_{p_0}D_0)^*$. Then define

$$\eta := (a \circ \vec{\alpha}) \times \left(b \circ \vec{\beta}\right) : I \longrightarrow S \times D_0^\delta.$$

And let γ be a path over $\vec{v}_0$ based at $\vec{w}_0$ representing an element in $\pi_1\left(Z_{\vec{v}_0}, \vec{w}_0\right)$. It is to show:

$$\mathcal{J} \circ \vec{\eta}_* \left([\gamma]\right) = \eta_* \circ \mathcal{J}\left([\gamma]\right).$$

Let $H : I \times I \to Z_0$ be a homotopy with the property that $H(\cdot, 0) = \gamma$ and $H(\cdot, s)$ is a path over $\vec{\alpha}(s)$ based at $\vec{\beta}(s)$ for every $s \in I$. Like the construction of $\overset{\circ}{\gamma}$ from γ on page 87 we construct a homotopy $\overset{\circ}{H}: I \times I \to \overset{\circ}{Z}$ from H such that $\overset{\circ}{H}(\cdot, 0) = \overset{\circ}{\gamma}$ and such that $\overset{\circ}{H}(\cdot, s)$ is a path $\overset{\circ}{Z}_{a\circ\vec{\alpha}(s)}$ based at $b \circ \vec{\beta}(s)$. This can be achieved by the definition:

$$\overset{\circ}{H}(\tau, s) := (H(\cdot, s))^\circ(\tau).$$

Now compute:

$$\mathcal{J} \circ \vec{\eta}_* \left([\gamma]\right) = \mathcal{J}\left([H(\cdot, 1)]\right) = \left[\overset{\circ}{H}(\cdot, 1)\right] \text{ and } \eta_* \circ \mathcal{J}\left([\gamma]\right) = \eta_*\left([\overset{\circ}{\gamma}]\right) = \left[\overset{\circ}{H}(\cdot, 1)\right].$$

□

Chapter 6

Line Integrals on the Nearby Fundamental Group

In Chapter 5 we defined the group

$$\pi_1\left(Z_{\vec{v}}, \vec{w}\right).$$

Consider the group ring $\mathbb{Z}\pi_1\left(Z_{\vec{v}}, \vec{w}\right)$ and let $J_{\vec{v}\vec{w}}$ or simply $\vec{J}$ be the augmentation ideal. By definition, this is the set of elements in $\mathbb{Z}\pi_1\left(Z_{\vec{v}}, \vec{w}\right)$, where the sum of the coefficients is vanishing.

The aim of the next two chapters is to put a $\mathbb{Z}$-MHS on

$$\mathbb{Z}\pi_1\left(Z_{\vec{v}}, \vec{w}\right) \Big/ \vec{J}^{s+1}$$

for any $s \geq 1$. This will be done by putting a MHS on $(\vec{J}/\vec{J}^{s+1})^*$. Note that we have the short exact sequence

$$0 \to \vec{J}^s/\vec{J}^{s+1} \to \vec{J}/\vec{J}^{s+1} \to \vec{J}/\vec{J}^s \to 0 \tag{6.1}$$

and that $\vec{J}^s/\vec{J}^{s+1} \cong (\vec{J}/\vec{J}^2)^{\otimes s}$ (recall that: $\pi_1\left(Z_{\vec{v}}, \vec{w}\right) \cong \pi_1\left(Z_t, p\right)$ for some $t \in \Delta^*$ and $p \in Z_t$).

The main ingredient for the definition of these MHS's is a differential graded algebra (dga) $A^\bullet$, which is made up from differential forms on the central fiber but whose cohomology is isomorphic to the cohomology of a regular fiber. It was suggested to us by Hain[1] to use a dga like $A^\bullet$ and he proposed the problem of finding a way to integrate elements of this dga somehow on the central fiber. We will see that the notion of *a path over $\vec{v}$ (based at $\vec{w}$)* will be the right notion of path for this purpose. We show that closed elements in A^1 can be integrated along paths in Z_0 over $\vec{v}$ in a natural way. This yields an integral structure on $H^1(A^\bullet)$, which is by this integration dual to $\vec{J}/\vec{J}^2$. On $A^\bullet$ we

[1]The dga $A^\bullet$ is in principle a modified version of the complex $P^\bullet_{\mathbb{C}}(X^*)[\theta]$ of [Hai87b] for this special situation with non compact fibers.

have filtrations $W_\bullet$ and $F^\bullet$ such that the induced filtrations together with the integral structure define a $\mathbb{Z}$-MHS on $H^1(A^\bullet)$. It will turn out that the integral lattice of this $\mathbb{Z}$-MHS depends on $\vec{v}$ (and not on $\vec{w}$). The dependence upon $\vec{v}$ is such that $\{(J_{\vec{v},\vec{w}}/J^2_{\vec{v},\vec{w}})^*\}_{\vec{v}\in(T_0\Delta)^*}$ is a nilpotent orbit of MHSs.

Since $A^\bullet$ has an algebra-structure, we can also define *iterated line integrals* along paths over $\vec{v}$. We shall do this in chapter 7. This will give a way to describe $(\vec{J}/\vec{J}^{s+1})^*$ in terms of the dga $A^\bullet$, similar to Chen's theorem in 1. Finally it will lead to the definition of a MHS on $(\vec{J}/\vec{J}^{s+1})^*$ such that the short exact sequence 6.1 becomes a short exact sequence of MHSs.

6.1 A Differential Graded Algebra of Differential Forms on the Central Fiber

Let $(\Lambda^\bullet, d) := (\bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})[\log t], d)$ be the dga, which is defined in the following way. Let $\bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})$ be the free graded-commutative $\mathbb{C}$-algebra with unit, generated by the symbols $\frac{dx}{x}$, $\frac{dy}{y}$ in degree 1.

Let then $\Lambda^\bullet := \bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})[\log t]$ be the algebra $\bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})$, where the symbol[2] $\log t$ or u has been added in degree 0. The differential d on $\Lambda^\bullet$ is defined by $d\left(\frac{dx}{x}\right) = d\left(\frac{dy}{y}\right) = 0$,

$$d\,(\log t) := \frac{dx}{x} + \frac{dy}{y}$$

and the Leibniz-rule. Note that $(\Lambda^\bullet, d)$ computes the cohomology of the fibers of the map $\Delta^* \times \Delta^* \to \Delta^*$ defined by $(x, y) \mapsto xy$.

For each component D_i of D^+ let

$$E^\bullet(D_i \log P_i) := \Gamma\left(\Omega^\bullet(D_i \log P_i) \otimes_{\Omega^0(D_i)} \mathcal{E}^{0,\bullet}(D_i)\right)$$

be the logarithmic de Rham complex. Here, $\mathcal{E}^{0,\bullet}(D_i)$ denotes the sheaf of C^∞-forms of type $(0, \bullet)$. Note that $(E^\bullet(D_i \log P_i), d)$ computes the cohomology of $D_i \setminus P_i$.

Finally let $\bigwedge^\bullet(\frac{dp}{p})$ be the differential graded-commutative $\mathbb{C}$-algebra with unit generated by the symbol $\frac{dp}{p}$ in degree 1, where the differential d is defined to be always the zero-map. Also $\left(\bigwedge^\bullet(\frac{dp}{p}), d\right)$ computes the cohomology of $\mathbb{C}^*$. We consider elements g_0 of $\bigwedge^0(\frac{dp}{p}) = \mathbb{C}$ as functions on the point p_0, i. e. $g_0(p_0) := g_0$. We refer to the complex number ρ of an element $\rho\frac{dp}{p} \in \bigwedge^1(\frac{dp}{p})$ as *the residue in* p_0 of this element. Write $\mathrm{Res}_{p_0}(\rho\frac{dp}{p}) = \rho$.

The dga $A^\bullet$, which we are going to define will be a sub-dga of

$$B^\bullet := \bigwedge^\bullet\left(\frac{dp}{p}\right) \oplus \bigoplus_{i>0} E^\bullet(D_i \log P_i) \oplus \bigoplus_{[k<l]} E^\bullet(\Delta^1) \otimes_{\mathbb{C}} \Lambda^\bullet,$$

[2] We will use u for convenience or if confusion with the function log might occur.

where Δ^1 denotes the one-dimensional standard-simplex (the unit interval) and $E^\bullet(\Delta^1) = \mathbb{C}[\xi] \oplus \mathbb{C}[\xi]d\xi$ the dga of real polynomial differential forms on Δ^1.

The elements of $A^\bullet$ in $B^\bullet$ will be called *the compatible elements* in $B^\bullet$. Hence, we will define $A^\bullet$ now by defining, when an element of $B^\bullet$ is compatible. Observe that $B^n = 0$ for $n \geq 4$.

We shall use P_{kl}, K_{kl}, L_{kl}, H_{kl}, R_{kl}, S_{kl}, T_{kl} and U_{kl} to denote elements in $\mathbb{C}[\xi, u]$. If the context is clear, we omit the indices k, l. Moreover, it will be useful to expand these elements of $\mathbb{C}[\xi, u]$ as polynomials in u with coefficients in $\mathbb{C}[\xi]$. That means for instance

$$P_{kl} = P_{kl}(\xi, u) = P = P_0 + P_1 u + P_2 u^2 + \cdots + P_m u^m \text{ and } P' = \frac{\partial P}{\partial \xi}.$$

$\mathbf{A^0}$: An element $f = \sum_{i \geq 0} g_i + \sum_{[k<l]} P_{kl} \in B^0$, where $g_0 \in \bigwedge^0(\frac{dp}{p})$ and $g_i \in E^0(D_i)$ for $i > 0$, is called *compatible* iff for $[k < l]$ holds:

$$P_{kl}(0, u) = g_k(p_{kl}) \text{ and } P_{kl}(1, u) = g_l(p_{kl}).$$

$\mathbf{A^1}$: We call an element in B^1,

$$\varphi = \sum_{i \geq 0} \omega_i + \sum_{[k<l]} \left(K_{kl} \frac{dx}{x} + L_{kl} \frac{dy}{y} + H_{kl} d\xi \right),$$

where $\omega_0 \in \bigwedge^1(\frac{dp}{p})$ and $\omega_i \in E^1(D_i \log P_i)$ for $i > 0$, a *compatible* element iff for $[k < l]$ holds:

$$\begin{array}{rcl} K_{kl}(0, u) = & \mathrm{Res}_{p_{kl}} \omega_k & , \quad L_{kl}(0, u) = 0, \\ K_{kl}(1, u) = & 0 & , \quad L_{kl}(1, u) = \mathrm{Res}_{p_{kl}} \omega_l. \end{array}$$

$\mathbf{A^2}$: Let us call an element

$$\phi = \sum_{i>0} \Omega_i + \sum_{[k<l]} \left(R_{kl}\, d\xi \wedge \frac{dx}{x} + S_{kl}\, d\xi \wedge \frac{dy}{y} + T_{kl} \frac{dx}{x} \wedge \frac{dy}{y} \right) \in B^2,$$

with $\Omega_i \in E^2(D_i \log P_i)$ for $i > 0$, *compatible* iff for $[k < l]$ holds:

$$T_{kl}(0, u) = T_{kl}(1, u) = 0.$$

$\mathbf{A^3}$: We define $A^3 := B^3$. That is, all elements of B^3 are *compatible.*

Observe that holds $dA^i \subset A^{i+1}$ for $i \geq 0$ and that $A^\bullet$ is a dga.

Definition 6.1 On $A^\bullet$ we define an *augmentation map* a by:

$$\begin{array}{rccc} a: & A^0 & \to & \mathbb{C} \\ & \sum_{i \geq 0} g_i + \sum_{[k<l]} P_{kl} & \mapsto & g_0. \end{array}$$

An alternative definition of $A^\bullet$ can be given as follows. Consider the surjective map of complexes

$$\Phi : B^\bullet \longrightarrow \bigoplus_{[k<l]} \Lambda^\bullet \oplus \Lambda^\bullet,$$

which sends $\sum_{i\geq 0} g_i + \sum_{[k<l]} P_{kl} \in B^0$ to

$$\sum_{[k<l]} \{(P_{kl}(0,u) - g_k(p_{kl})) \oplus (P_{kl}(1,u) - g_l(p_{kl}))\}$$

and $\sum_{i\geq 0} \omega_i + \sum_{[k<l]} \left(K_{kl}\frac{dx}{x} + L_{kl}\frac{dy}{y} + H_{kl}d\xi\right) \in B^1$ to

$$\sum_{[k<l]} \left\{ \left[(K_{kl}(0,u) - \mathrm{Res}_{p_{kl}}\omega_k) \frac{dx}{x} + L_{kl}(0,u)\frac{dy}{y} \right] \right.$$
$$\left. \oplus \quad \left[K_{kl}(1,u)\frac{dx}{x} + (L_{kl}(1,u) - \mathrm{Res}_{p_{kl}}\omega_l) \frac{dy}{y} \right] \right\}.$$

Moreover, Φ maps

$$\sum_{i>0} \Omega_i + \sum_{[k<l]} R_{kl}\, d\xi \wedge \frac{dx}{x} + S_{kl}\, d\xi \wedge \frac{dy}{y} + T_{kl}\, \frac{dx}{x} \wedge \frac{dy}{y} \in B^2$$

to $\sum_{[k<l]} T_{kl}(0,u)\frac{dx}{x} \wedge \frac{dy}{y} \oplus T_{kl}(1,u)\frac{dx}{x} \wedge \frac{dy}{y}$. It is easy to check that Φ is indeed a surjective map of complexes. Then we find:

$$A^\bullet = \ker \Phi.$$

Define $C^\bullet := \sum_{[k<l]} \Lambda^\bullet \oplus \Lambda^\bullet$ and note that we have the short exact sequence

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0,$$

which yields a long exact cohomology sequence (as $H^i(B^\bullet) = 0$ for $i \geq 2$)

$$\begin{aligned} 0 \to H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet) & \\ \to H^1(A^\bullet) \to H^1(B^\bullet) \to H^1(C^\bullet) \to H^2(A^\bullet) \to 0. & \quad (6.2) \end{aligned}$$

Theorem 6.2 *$H^\bullet(A^\bullet) \cong H^\bullet(Z_t; \mathbb{C})$ for any $t \in \Delta^*$.*

Proof: The only closed elements in A^0 are the constant compatible functions. Therefore: $H^0(A^\bullet) \cong H^0(Z_t; \mathbb{C}) \cong \mathbb{C}$.

We make use of the fact that the map $\mathring{h}:\mathring{Z}\to\mathring{\Delta}$ is a locally trivial fibration and prove: $H^\bullet(A^\bullet) \cong H^\bullet(\mathring{Z}_{(0,1)}; \mathbb{C})$.

Recall that we can build up $\overset{\circ}{Z}_{(0,1)}$ in the following way. For each p_{kl} with $[k < l]$ we have the open disks $U^k_{kl} \subset D_k$ and $U^l_{kl} \subset D_l$ around the double point p_{kl}. Now define

$$B_{kl} := \left(\overset{\circ}{U}{}^k_{kl} \amalg \overset{\circ}{U}{}^l_{kl}\right) \Big/ \sim \quad ,$$

where "$\sim$" denotes the equivalence relation:

$$\begin{array}{rcl} (R_1, e^{2\pi\imath\phi_1}) & \sim & (R_2, e^{2\pi\imath\phi_2}) \\ :\Leftrightarrow R_1 = R_2 = 0 & \text{and} & \phi_1 = -\phi_2. \end{array}$$

Then $U^{k\,*}_{kl} \amalg U^{l\,*}_{kl}$ can be considered as subspace of B_{kl} and as subspace of $\coprod_{\imath\geq 0} D^*_\imath = \coprod_{\imath\geq 0} D_\imath \setminus P_\imath$. We construct the fiber $\overset{\circ}{Z}_{(0,1)}$ by glueing $\coprod_{[k<l]} B_{kl}$ and $\coprod_{\imath\geq 0} D^*_\imath$ together along the $U^{k\,*}_{kl} \amalg U^{l\,*}_{kl}$'s. Remark that

$$\left(\coprod_{\imath\geq 0} D^*_\imath, \coprod_{[k<l]} B_{kl}\right)$$

is an excisive couple (cf. [Spa66] p. 188) for the space $\overset{\circ}{Z}_{(0,1)}$, which is to say that we may apply Mayer-Vietoris. Hence, there is an exact sequence

$$\begin{aligned} 0 \to H^0(\overset{\circ}{Z}_{(0,1)}) \to \bigoplus_{\imath\geq 0} H^0(D^*_\imath) \oplus \bigoplus_{[k<l]} H^0(B_{kl}) \to \bigoplus_{[k<l]} H^0(U^{k\,*}_{kl}) \oplus H^0(U^{k\,*}_{kl}) \\ \to H^1(\overset{\circ}{Z}_{(0,1)}) \to \bigoplus_{\imath\geq 0} H^1(D^*_\imath) \oplus \bigoplus_{[k<l]} H^1(B_{kl}) \to \bigoplus_{[k<l]} H^1(U^{k\,*}_{kl}) \oplus H^1(U^{k\,*}_{kl}) \to 0. \end{aligned} \tag{6.3}$$

Now we compare the last terms in (6.3) with the last terms in (6.2). That is, we define isomorphisms φ, ψ such that

$$\begin{array}{ccccc} H^1(B^\bullet) & \xrightarrow{\Phi} & H^1(C^\bullet) & & \to H^2(A^\bullet) \to 0 \\ \cong \downarrow\varphi & & \cong\downarrow\psi & & \downarrow \\ \bigoplus_{\imath\geq 0} H^1(D^*_\imath) \oplus \bigoplus_{[k<l]} H^1(B_{kl}) & \to & \bigoplus_{[k<l]} H^1(U^{k\,*}_{kl}) \oplus H^1(U^{k\,*}_{kl}) & \to & 0 \end{array}$$

commutes. It is then a consequence of the 5-lemma that $H^2(A^\bullet) = 0$.

Denote by $\{\frac{dx}{x}\}$ (resp. $\{\frac{dy}{y}\}$) the cohomology class in $H^1(B_{kl})$, which has value $2\pi i$ on the positive generator of $H_1(U^{k\,*}_{kl})$ (resp. $H_1(U^{l\,*}_{kl})$) in $H_1(B_{kl})$. Then $\{\frac{dx}{x}\} = -\{\frac{dy}{y}\}$.

We define an isomorphism

$$\begin{array}{lccc} \varphi: & H^1(B^\bullet) & \to & \bigoplus_{\imath\geq 0} H^1(D^*_\imath) \oplus \bigoplus_{[k<l]} H^1(B_{kl}) \\ & [\sum_{\imath\geq 0} \omega_\imath + \sum_{[k<l]} K\frac{dx}{x} + L\frac{dy}{y} + Hd\xi] & \mapsto & \sum_{\imath\geq 0} [\omega_\imath] + \sum_{[k<l]} K\{\frac{dx}{x}\} + L\{\frac{dy}{y}\} \end{array}$$

(Note that for closed $\sum_{\imath\geq 0}\omega_\imath + \sum_{[k<l]} K\frac{dx}{x} + L\frac{dy}{y} + Hd\xi$, the polynomial $K(\xi,u) - L(\xi,u)$ is constant, i. e. $K(\xi,u) - L(\xi,u) = K(0,0) - L(0,0) = K(1,0) - L(1,0)$. Therefore, $K\{\frac{dx}{x}\} + L\{\frac{dy}{y}\} = (K-L)\{\frac{dx}{x}\}$ makes sense.)

Define moreover the isomorphism

$$\begin{array}{llll} \psi: & H^1(C^\bullet) & \to & \bigoplus_{[k<l]} H^1(U_{kl}^{k\,*}) \oplus H^1(U_{kl}^{k\,*}) \\ \multicolumn{2}{l}{\sum_{[k<l]} (a_k\frac{dx}{x} + b_k\frac{dy}{y}) + (a_l\frac{dx}{x} + b_l\frac{dy}{y})} & \mapsto & \sum_{[k<l]} (a_k - b_k)\{\frac{dx}{x}\} + (b_l - a_l)\{\frac{dy}{y}\} \end{array}$$

and note that the above diagram commutes.

Finally, if we use the fact that the Euler characteristics in (6.2) and (6.3) are zero, we find:

$$\begin{aligned} \dim_{\mathbb{C}} H^1(A^\bullet) = \dim_{\mathbb{C}} H^1(\mathring{Z}_{(0,1)}) &= 1 + \sum_{\imath\geq 0} \left(\dim_{\mathbb{C}} H^1(D_\imath^*) - 1\right) \\ &= 2 + \sum_{\imath\geq 0} \left(\dim_{\mathbb{C}} H^1(D_\imath) - 2\right) + \sum_{[k<l]} 2. \quad \square \end{aligned}$$

6.2 Integration in the Nearby Fiber along Paths over a Tangent Vector

Here we are going to define integrals of a closed element

$$\varphi = \sum_{\imath\geq 0}\omega_\imath + \sum_{[k<l]} \left(K_{kl}\frac{dx}{x} + L_{kl}\frac{dy}{y} + H_{kl}d\xi\right) \in A^1$$

along paths over $\vec{v}$. We define those integrals in four steps by defining them first in special cases.

(a) Let $\gamma : [a;b] \to Z_0$ be a path over $\vec{v}$, which meets the set of double points only once with parameter value $\tau_0 \in]a;b[$, where it changes from D_k to D_l. Let $(x,y) : W_{kl} = U_{kl}^k \times U_{kl}^l \to \mathbb{C}^2$ and $t : \Delta \to \mathbb{C}$ be coordinates such that $h(x,y) = x \cdot y$ and $\vec{v} = \frac{\partial}{\partial t}$. Define $\gamma_x(\tau)$ (resp. $\gamma_y(\tau)$) to be $x(\gamma(\tau))$ (resp. $y(\gamma(\tau))$) for all $\tau \in [a;b]$ with $\gamma(\tau) \in U_{kl}^k$ (resp. $\gamma(\tau) \in U_{kl}^l$). Then define:

$$\int_\gamma \varphi := \lim_{\varepsilon\to 0} \left\{ \int_{\gamma\leq\tau_0-\varepsilon} \omega_k + \int_0^1 H\left(\xi, \log\left(\gamma_x(\tau_0-\varepsilon)\gamma_y(\tau_0+\varepsilon)\right)\right) d\xi + \int_{\gamma\geq\tau_0+\varepsilon} \omega_l \right\}.$$

(b) Let $\gamma : [a;b] \to Z_0$ be a path over $\vec{v}$, which meets the set of double points only once with parameter value $\tau_0 \in]a;b[$, where it stays in one component D_k. Then define:

$$\int_\gamma \varphi := \lim_{\varepsilon\to 0} \left\{ \int_{\gamma\leq\tau_0-\varepsilon} \omega_k + \int_{\gamma\geq\tau_0+\varepsilon} \omega_k \right\}.$$

(c) Now let $\gamma : [a;b] \to Z_0$ be a path over $\vec{v}$ such that $\gamma(a)$ and $\gamma(b)$ are no double points. Then there is a finite number of parameter values $\tau_1, \dots, \tau_N$ with $a < \tau_1 < \dots < \tau_N < b$, which are mapped onto double points. Choose a $\tau_i^* \in]\tau_i; \tau_{i+1}[$ for $i = 1, \dots, N-1$ and let $\tau_0^* := a$ and $\tau_N^* := b$. Then define for $i = 1, \dots, N$ the paths $\gamma_i := \gamma|_{[\tau_{i-1}^*, \tau_i^*]}$ and we define:

$$\int_\gamma \varphi := \sum_{i=1}^{N} \int_{\gamma_i} \varphi.$$

(d) Finally, let $\gamma : [a;b] \to Z_0$ be a path in Z_0 over $\vec{v}$ based at $\vec{w}$. The coordinate $p : D_0 \to \mathbb{C}$ allows us to consider D_0 as part of the complex plane. There, in the complex plane, we have the differential form (and not the symbol) $\frac{dp}{p}$. Let σ be the straight path $\tau \mapsto (1-\tau)$ in $\mathbb{C}$, then we define

$$\int_\gamma \varphi := \int_{\sigma \star \gamma \star \sigma^{-1}} \varphi.$$

Theorem 6.3 *For a path $\gamma : [a;b] \to Z_0$ over $\vec{v}$ based at $\vec{w}$ and a closed $\varphi \in A^1$ holds:*

(i) $\int_\gamma \varphi$ is well-defined and finite,

(ii) $\int_\gamma \varphi$ does not depend on the choice of coordinates,

(iii) $\int_\gamma \varphi = 0$, if φ is exact.

Lemma 6.4 *Any closed $\omega \in E^1(D_i \log P_i)$ can locally on a coordinate (U, z) on D_i around a double point p_{kl} be written as*

$$\omega = Res_{p_{kl}} \omega \cdot \frac{dz}{z} + \psi,$$

where ψ is a smooth 1-form in $E^1(U)$.

The proof of this lemma is analogous to the proof of Lemma 3.7. □

Proof of 6.3: Let $\varphi = \sum_{i \geq 0} \omega_i + \sum_{[k<l]} K_{kl} \frac{dx}{x} + L_{kl} \frac{dy}{y} + H_{kl} d\xi$. To prove (i), we have to show that integrals of type (a) and (b) are well-defined, finite and do not depend on the choice of coordinates (note that (d) is similar to (b)). Assume that we are given coordinates x, y and t like in (a) and write

$$\omega_k = \mathrm{Res}_{p_{kl}} \omega_k \frac{dx}{x} + \psi_k \quad \text{and} \quad \omega_l = \mathrm{Res}_{p_{kl}} \omega_l \frac{dx}{x} + \psi_l.$$

ad (a) That φ is closed implies $K' = \frac{\partial H}{\partial u}$, $L' = \frac{\partial H}{\partial u}$ and $\frac{\partial L}{\partial u} = \frac{\partial K}{\partial u}$. Let $H = H_0 + H_1 u + \dots + H_m u^m$. Then $iH_i = K'_{i-1} = L'_{i-1}$ for $i = 1, \dots, m$. Since $K(0,u)$, $K(1,u)$, $L(0,u)$, $L(1,u)$ are constants in $\mathbb{C}$ we have $K_i(0) = K_i(1) =$

$L_i(0) = L_i(1) = 0$ for $i \geq 1$. Moreover $\mathrm{Res}_{p_{kl}}\omega_k = -\mathrm{Res}_{p_{kl}}\omega_l$ and because of $H_1 = K_0' = L_0'$ we obtain $\int_0^1 H_1(\xi)d\xi = \mathrm{Res}_{p_{kl}}\omega_l$. Therefore we derived:

$$\int_0^1 H(\xi, u)d\xi = \int_0^1 H_0(\xi)d\xi + \mathrm{Res}_{p_{kl}}\omega_l \cdot u.$$

Now we may compute as follows: recall $\gamma(\tau_0) = p_{kl}$ and choose ε_0 such that $\gamma([\tau_0 - \varepsilon_0; \tau_0 + \varepsilon_0]) \subset W_{kl}$ and $\gamma_x([\tau_0 - \varepsilon_0; \tau_0])$ as well as $\gamma_y([\tau_0; \tau_0 + \varepsilon_0])$ lie within an open set of $\mathbb{C}$, where the logarithm is univalent. Then we find, when we split up $\omega_k = \mathrm{Res}_{p_{kl}}\,\omega_k \frac{dx}{x} + \psi_k$ and $\omega_l = \mathrm{Res}_{p_{kl}}\,\omega_l \frac{dy}{y} + \psi_l$:

$$\int_\gamma \varphi = \lim_{\varepsilon \to 0} \left\{ \int_{\gamma \leq \tau_0 - \varepsilon} \omega_k + \int_0^1 H\left(\xi, \log\left(\gamma_x(\tau_0 - \varepsilon)\gamma_y(\tau_0 + \varepsilon)\right)\right) d\xi + \int_{\gamma \geq \tau_0 + \varepsilon} \omega_l \right\}$$

$$= \int_{\gamma \leq \tau_0 - \varepsilon_0} \omega_k + \int_{\gamma {\leq \tau_0 \atop \geq \tau_0 - \varepsilon_0}} \psi_k + \int_{\gamma \geq \tau_0 + \varepsilon_0} \omega_l + \int_{\gamma {\leq \tau_0 + \varepsilon_0 \atop \geq \tau_0}} \psi_l + \int_0^1 H_0(\xi)d\xi$$

$$+ \lim_{\varepsilon \to 0} \left\{ \int_{\gamma {\leq \tau_0 - \varepsilon \atop \geq \tau_0 - \varepsilon_0}} \mathrm{Res}_{p_{kl}}\omega_k \frac{dx}{x} + \mathrm{Res}_{p_{kl}}\,\omega_l \log\left(\gamma_x(\tau_0 - \varepsilon)\gamma_y(\tau_0 + \varepsilon)\right) + \int_{\gamma {\leq \tau + \varepsilon_0 \atop \geq \tau_0 + \varepsilon}} \mathrm{Res}_{p_{kl}}\omega_l \frac{dy}{y} \right\}.$$

And then remark that

$$\lim_{\varepsilon \to 0} \left\{ \int_{\gamma {\leq \tau_0 - \varepsilon \atop \geq \tau_0 + \varepsilon_0}} \frac{dx}{x} - \log\left(\gamma_x(\tau_0 - \varepsilon)\right) + \int_{\gamma {\leq \tau_0 + \varepsilon_0 \atop \geq \tau_0 + \varepsilon}} -\frac{dy}{y} + \log\left(\gamma_y(\tau_0 + \varepsilon)\right) \right\}$$
$$= -\log\left(\gamma_x(\tau_0 - \varepsilon_0)\right) + \log\left(\gamma_y(\tau_0 + \varepsilon_0)\right).$$

This shows that the integral in (a) is finite.

Once we know that $\int_\gamma \varphi$ converges, we may write $\int_\gamma \varphi$ as

$$\int_0^1 H_0 d\xi + \lim_{\varepsilon \to 0} \left\{ \int_{\gamma \leq \tau_0 - \varepsilon} \omega_k + \mathrm{Res}_{p_{kl}}\omega_l \log\left(\gamma_x(\tau_0 - \varepsilon)\gamma_y(\tau_0 + \varepsilon)\right) + \int_{\gamma \geq \tau_0 + \varepsilon} \omega_l \right\}$$

$$= \int_0^1 H_0 d\xi + \lim_{\varepsilon \to 0} \left\{ \int_{\gamma \leq \tau_0 - \varepsilon} \omega_k + \mathrm{Res}_{p_{kl}}\omega_l \log\left(\varepsilon^2\right) + \int_{\gamma \geq \tau_0 + \varepsilon} \omega_l \right.$$
$$\left. + \mathrm{Res}_{p_{kl}}\omega_l \log\left(\frac{\gamma_x(\tau_0 - \varepsilon)}{\varepsilon} \cdot \frac{\gamma_y(\tau_0 - \varepsilon)}{\varepsilon}\right) \right\}.$$

Observe that

$$\lim_{\varepsilon\to 0}\left\{\frac{\gamma_x(\tau_0-\varepsilon)}{\varepsilon}\cdot\frac{\gamma_y(\tau_0+\varepsilon)}{\varepsilon}\right\} = -\dot\gamma_x^{\le\tau_0}\cdot\dot\gamma_{y\,\ge\tau_0}$$

and

$$-\dot\gamma_x^{\le\tau_0}\cdot\dot\gamma_{y\,\ge\tau_0}\frac{\partial}{\partial t} = \langle -\dot\gamma_{\le\tau_0},\dot\gamma_{\ge\tau_0}\rangle = \frac{\partial}{\partial t}.$$

Hence, the second summand vanishes, i. e. :

$$\int_\gamma \varphi = \int_0^1 H_0 d\xi + \lim_{\varepsilon\to 0}\left\{\int_{\gamma^{\le\tau_0-\varepsilon}}\omega_k + \mathrm{Res}_{p_{kl}}\omega_l \log\left(\varepsilon^2\right) + \int_{\gamma_{\ge\tau_0+\varepsilon}}\omega_l\right\}.$$

This expression obviously does not depend upon the choice of the coordinates (x,y) and t.

If $\varphi = d\left(\sum_{\imath\ge 0} g_\imath + \sum_{[k<l]} P_{kl}\right) = \sum_{\imath\ge 0} dg_\imath + \sum_{[k<l]}\frac{\partial P_{kl}}{\partial u}\left(\frac{dx}{x}+\frac{dy}{y}\right) + P'_{kl}\,d\xi$, then

$$\begin{aligned}\int_\gamma\varphi &= P_{kl}(1,u) - P_{kl}(0,u)\\ &+ \lim_{\varepsilon\to 0}\left\{g_k(\gamma(\tau_0-\varepsilon)) - g_k(\gamma(0)) + g_l(\gamma(1)) - g_l(\gamma(\tau_0+\varepsilon))\right\}\\ &= g_l(\gamma(1)) - g_k(\gamma(0)).\end{aligned}$$

ad (b) The definition does not depend on the choice of coordinates. We just have to show that the integral converges.

$$\begin{aligned}\int_\gamma\varphi = \int_{\gamma^{\le\tau_0-\varepsilon_0}}\omega_k + \int_{\gamma^{\le\tau_0}_{\ge\tau_0-\varepsilon_0}}\psi_k + \int_{\gamma_{\ge\tau_0+\varepsilon_0}}\omega_l + \int_{\gamma^{\le\tau_0+\varepsilon_0}_{\ge\tau_0}}\psi_l\\ + \mathrm{Res}_{p_{kl}}\,\omega_k \lim_{\varepsilon\to 0}\left\{\int_{\gamma^{\le\tau_0-\varepsilon}_{\ge\tau_0-\varepsilon_0}}\frac{dx}{x} + \int_{\gamma^{\le\tau_0+\varepsilon_0}_{\ge\tau_0+\varepsilon}}\frac{dx}{x}\right\}.\end{aligned}$$

And further

$$\lim_{\varepsilon\to 0}\left\{\int_{\gamma^{\le\tau_0-\varepsilon}_{\ge\tau_0-\varepsilon_0}}\frac{dx}{x} + \int_{\gamma^{\le\tau_0+\varepsilon_0}_{\ge\tau_0+\varepsilon}}\frac{dx}{x}\right\}$$

$$= \log(\gamma_x(\tau_0+\varepsilon_0)) - \log(\gamma_x(\tau_0-\varepsilon_0)) + \lim_{\varepsilon\to 0}\left(\log(\gamma_x(\tau_0-\varepsilon)) - \log(\gamma_x(\tau_0+\varepsilon))\right).$$

Now is

$$\begin{aligned}
& \lim_{\varepsilon \to 0} \{\log(\gamma_x(\tau_0 - \varepsilon)) - \log(\gamma_x(\tau_0 + \varepsilon))\} \\
= & \lim_{\varepsilon \to 0} \left\{ \log\left(\frac{\gamma_x(\tau_0 - \varepsilon)}{\varepsilon}\right) - \log\left(\frac{\gamma_x(\tau_0 + \varepsilon)}{\varepsilon}\right)\right\} \\
= & \log\left(-\dot{\gamma}_x^{\leq \tau_0}\right) - \log\left(\dot{\gamma}_{x\ \geq \tau_0}\right) \\
= & \ 0 \quad \text{by the definition of a path over } \vec{v}.
\end{aligned}$$

If φ is exact, we have $\int_\gamma \varphi = g_k(\gamma(1)) - g_k(\gamma(0))$. □

Remark 6.5 We see from the proof of Theorem 6.3 that we have in the case of an integral of type (a) like on page 94:

$$\begin{aligned}
\int_\gamma \varphi = \int_0^1 H_0 d\xi + \lim_{\varepsilon \to 0} & \left\{ \int_{\gamma \leq \tau_0 - \varepsilon} \omega_k + \mathrm{Res}_{p_{kl}} \omega_l \log\left(\varepsilon^2\right) + \int_{\gamma \geq \tau_0 + \varepsilon} \omega_l \right\}, \\
= \ & \lim_{\varepsilon \to 0} \left\{ \int_{\gamma \leq \tau_0 - \varepsilon} \omega_k - \mathrm{Res}_{p_{kl}} \omega_k \log\left(\gamma_x(\tau_0 - \varepsilon)\right) \right\} \\
+ \ & \lim_{\varepsilon \to 0} \left\{ \int_{\gamma \geq \tau_0 + \varepsilon} \omega_l + \mathrm{Res}_{p_{kl}} \omega_l \log\left(\gamma_y(\tau_0 + \varepsilon)\right) \right\} \\
+ \ & \int_0^1 H_{kl}(\xi, 0) d\xi.
\end{aligned}$$

Remark 6.6 Observe that if one considers in (a) the closed

$$\Xi := K\frac{dx}{x} + L\frac{dy}{y} + H\, d\xi \in \left(\bigwedge^\bullet\left(\frac{dx}{x}, \frac{dy}{y}\right)[\log t] \otimes E^\bullet(\Delta^1)\right)^1$$

as element in

$$\left(\bigwedge^\bullet\left(\frac{dx}{x}, \frac{dy}{y}\right)[\log x, \log y] \otimes E^\bullet(\Delta^1)\right)^1,$$

which is defined in the obvious way (for the precise definition see p. 114), by setting

$$d\log x = \frac{dx}{x} \qquad \text{and} \qquad d\log y = \frac{dy}{y},$$

then it is even exact. If $P \in \mathbb{C}[\xi, \log x, \log y]$ is such that $dP = \Xi$, then

$$\int_{[0;1]} H(\xi, \log t) d\xi = P(1, \log x, \log y) - P(0, \log x, \log y).$$

Next we state that $\int_\gamma \varphi$ does not depend on the choice of a path γ within a homotopy class in $\pi_1(Z_{\vec{v}}, \vec{w})$.

Theorem 6.7 *Let $\varphi \in A^1$ be closed and let γ_0, $\gamma_1 : I \to Z_0$ be two paths over $\vec{v}$ based at $\vec{w}$, which are nearby homotopic. Then*

$$\int_{\gamma_0} \varphi = \int_{\gamma_1} \varphi.$$

We will prove the corresponding theorem for iterated integrals in general as Theorem 7.15. The Theorem 6.7 above is then just a special case of 7.15.

Theorem 6.8 *The integration of closed elements in A^1 along paths over $\vec{v}$ based at $\vec{w}$ defines an isomorphism*

$$H^1(A^\bullet) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}\left(\vec{J}/\vec{J}^2; \mathbb{C}\right).$$

Proof: Since we know that both $\mathbb{C}$-vector spaces have the same dimension, it suffices to show that the integration map is injective. Note that

$$\mathrm{Hom}_{\mathbb{Z}}\left(\vec{J}/\vec{J}^2; \mathbb{C}\right) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(\vec{J}; \mathbb{C}\right) \cong \mathrm{Hom}_{\mathbb{Z}}\left(\pi_1(Z_{\vec{v}}, \vec{w}); \mathbb{C}\right).$$

Assume $\varphi = \sum_{\imath \geq 0} \omega_\imath + \sum_{[k<l]} K_{kl}\frac{dx}{x} + L_{kl}\frac{dy}{y} + H_{kl}d\xi \in A^1$ is closed and that

$$\int_\gamma \varphi = 0 \qquad \text{for any } [\gamma] \in \pi_1(Z_{\vec{v}}, \vec{w}).$$

We claim that φ is exact. First, observe that all residues of φ are necessarily zero, as one sees by integration along a path over $\vec{v}$ based at $\vec{w}$, which approaches a double point, turns once around this double point and returns as it came. Moreover, all the $\omega_\imath$ are exact since the expression

$$g_\imath(p) := \int_{\vec{w}}^{p} \varphi \qquad\qquad \text{for } p \in D_\imath$$

makes sense and $dg_\imath = \omega_\imath$. Now define

$$f := \sum_{\imath \geq 0} g_\imath + \sum_{[k<l]} \int_0^\xi H(\xi, u)d\xi \in B^0$$

and note that it satisfies the compatibility condition and that $df = \varphi$. □

The Theorem 6.8 allows us to define an integral lattice $H^1(A^\bullet)_{\mathbb{Z}}$ within $H^1(A^\bullet)$ in the following way.

Definition 6.9 *Define $H^1(A^\bullet)_{\mathbb{Z}}$ to be the lattice of those cohomology classes which correspond to* $\mathrm{Hom}_{\mathbb{Z}}\left(\vec{J}/\vec{J}^2; \mathbb{Z}\right)$ *via the integration map. $H^1(A^\bullet)_{\mathbb{Q}}$ is then defined as*

$$H^1(A^\bullet)_{\mathbb{Q}} := \mathrm{Hom}_{\mathbb{Z}}\left(\vec{J}/\vec{J}^2; \mathbb{Q}\right) \subset H^1(A^\bullet).$$

Remark: Note that in contrast to $H^1(A^\bullet)$ the lattice $H^1(A^\bullet)_{\mathbb{Z}}$ depends a priori on the tangent vectors $\vec{v}$ and $\vec{w}$.

6.3 The Hodge- and Weight Filtration

Next we define a decreasing filtration $F^\bullet$ and an increasing filtration $W_\bullet$ on $A^\bullet$ by defining such filtrations on $B^\bullet$.

On $\Lambda^\bullet$ we set

$$F^p\Lambda^\bullet := \bigoplus_{n+m\geq p} \bigwedge^n \left(\frac{dx}{x}, \frac{dy}{y}\right)(\log t)^m$$

and

$$W_l\Lambda^\bullet := \bigoplus_{n+2\cdot m\leq l} \bigwedge^n \left(\frac{dx}{x}, \frac{dy}{y}\right)(\log t)^m.$$

In particular we have: $\log t \in W_2 \cap F^1$.

Then define on $E^\bullet(\Delta^1) \otimes \Lambda^\bullet$ the filtrations $F^\bullet$ and $W_\bullet$ by:

$$\begin{aligned} F^p\left(E^\bullet(\Delta^1) \otimes \Lambda^\bullet\right) &:= E^\bullet(\Delta^1) \otimes F^p\Lambda^\bullet \\ W_l\left(E^\alpha(\Delta^1) \otimes \Lambda^\beta\right) &:= E^\alpha(\Delta^1) \otimes W_{l+\alpha}\Lambda^\beta. \end{aligned}$$

On $E^\bullet(D_i \log P_i)$ for $i > 0$ we have the classical Hodge filtration:

$$F^pE^\bullet(D_i \log P_i) := \bigoplus_{n\geq p} \Gamma\left(\Omega^n(D_i \log P_i) \otimes_{\Omega^0(D_i)} \mathcal{E}^{0,\bullet-n}(D_i)\right)$$

and the weight filtration $W_\bullet$. This filtration $W_\bullet$ is multiplicative, i. e. $W_m \wedge W_n = W_{m+n}$, and hence determined by:

$$\begin{array}{rlrl} W_{-1}E^0(D_i \log P_i) = & 0; & W_{-1}E^1(D_i \log P_i) = & 0 \\ W_0\, E^0(D_i \log P_i) = & E^0(D_i); & W_0\, E^1(D_i \log P_i) = & E^1(D_i) \\ W_1\, E^0(D_i \log P_i) = & E^0(D_i); & W_1\, E^1(D_i \log P_i) = & E^1(D_i \log P_i). \end{array}$$

Finally, define on $\bigwedge^\cdot(\frac{dp}{p})$:

$$W_l \bigwedge^\bullet \left(\frac{dp}{p}\right) := \bigoplus_{n\leq l} \bigwedge^n \left(\frac{dp}{p}\right)$$

and

$$F^p \bigwedge^\bullet (\frac{dp}{p}) := \bigoplus_{n \geq p} \bigwedge^n (\frac{dp}{p}).$$

Putting all these filtrations together, we obtain filtrations $W_\bullet$ and $F^\bullet$ on $A^\bullet$. In particular, $W_\bullet$ and $F^\bullet$ allow us to define filtrations on the cohomology groups in the following way.

Definition 6.10 Now define the weight filtration $W_\bullet$ on $H^\bullet(A^\bullet)$ by:

$$W_{l+m} H^m(A^\bullet) := \operatorname{im}\{H^m(W_l A^\bullet) \to H^m(A^\bullet)\}.$$

And define the Hodge filtration $F^\bullet$ on $H^\bullet(A^\bullet)$ as:

$$F^p H^m(A^\bullet) := \operatorname{im}\{H^m(F^p A^\bullet) \to H^m(A^\bullet)\}.$$

On the $\mathbb{Q}$-vector space $H^1(A^\bullet)_\mathbb{Q}$, the weight filtration $W_\bullet$ on $H^1(A^\bullet)$ induces the filtration:

$$W_l H^1(A^\bullet)_\mathbb{Q} := W_l H^1(A^\bullet) \cap H^1(A^\bullet)_\mathbb{Q}.$$

Observe that

$$Gr_l^W \left(H^1(A^\bullet)_\mathbb{Q}\right) = \operatorname{im}\left\{W_l H^1(A^\bullet)_\mathbb{Q} \to Gr_l^W H^1(A^\bullet)\right\}.$$

We want to finish this section now by proving the following theorem.

Theorem 6.11

$$(J_{\vec{v}\vec{w}}/J_{\vec{v}\vec{w}}^2)^* := \left(H^1(A^\bullet)_\mathbb{Z},\ (H^1(A^\bullet)_\mathbb{Q}, W_\bullet),\ (H^1(A^\bullet), W_\bullet, F^\bullet)\right)$$

is a $\mathbb{Z}$-MHS of possible weights 0, 1, and 2.

Remark 6.12 If one computes the weight spectral sequence $(A^\bullet,\ W_\bullet)$ one finds infinitely many E_1-terms, which are nonzero, each of which is an infinite dimensional pure Hodge structure. Nevertheless, the spectral sequence degenerates at E_2. In the language of [Hai87a] the E_1-term is the image under a certain De Rham functor applied to the finite dimensional E_1-terms of the spectral sequences of $(E^\bullet(D_i \log P_i), W_\bullet)$ for $i > 0$, of $(\bigwedge^\bullet(\frac{dp}{p}), W_\bullet)$ and of $\Lambda^\bullet, W_\bullet)$ (cf. [Hai87a], the proof of (5.5.1)).

However, if one just wants to prove that $H^1(A^\bullet)$ is a MHS, the weight spectral sequence of $(A^\bullet,\ W_\bullet)$ suggests a shortcut, which avoids going all the way through the spectral sequences. This is how we will prove 6.11.

Furthermore, one could consider $(A^\bullet,\ W_\bullet,\ F^\bullet)$ as part of a mixed Hodge complex, which takes the integral structure (coming from integration along paths over $\vec{v}$) into account. We omit that construction, since we do not need it for our further investigations.

Proof: Since the category of MHSs is abelian, we can prove Theorem 6.11 if we show that there are exact sequences

(i) $0 \to Gr_2^W H^1(A^\bullet) \xrightarrow{\alpha} \bigoplus_{[k<l]} H^0(p_{kl})(-1) \xrightarrow{\beta} \bigoplus_{i>0} H^2(D_i) \to 0,$

(ii) $0 \to Gr_1^W H^1(A^\bullet) \xrightarrow{\cong} \bigoplus_{i>0} H^1(D_i) \to 0,$

(iii) $\bigoplus_{i\geq 0} H^0(D_i) \xrightarrow{\gamma} \bigoplus_{[k<l]} H^0(p_{kl}) \xrightarrow{\delta} Gr_0^W H^1(A^\bullet) \to 0,$

where all the maps are defined over $\mathbb{Q}$ and are strict with respect to the respective Hodge filtrations. $Gr_l^W H^1(A^\bullet)$ is then a Hodge structure of weight l since all the other terms are Hodge structures of appropriate weights.

Note that if we are given a $[\varphi] = [\sum_{i\geq 0} \omega_i + \sum_{[k<l]} K_{kl}\frac{dx}{x} + L_{kl}\frac{dy}{y} + H_{kl}d\xi] \in H^1(A^\bullet)$, then we can always write with complex numbers ρ_{kl}, $c_{kl} \in \mathbb{C}$

$$[\varphi] = [\sum_{i\geq 0} \omega_i + \sum_{[k<l]} \frac{1}{2\pi i}\rho_{kl}\,\Theta + c_{kl}d\xi],$$

where $\Theta := (1-\xi)\frac{dx}{x} - \xi\frac{dy}{y} - \log t\, d\xi \in W_1\left[(E^\bullet \otimes \Lambda^\bullet)^1\right]$.

This shows that $W_2 H^1(A^\bullet) = H^1(A^\bullet)$ and

$$[\varphi] \in W_1H^1(A^\bullet) \Leftrightarrow \forall [k<l]:\ \rho_{kl} = 0 \text{ and } [\varphi] \in W_0H^1(A^\bullet) \Leftrightarrow \forall i \geq 0:\ \omega_i = 0.$$

ad (i): We identify $H^0(p_{kl})(-1)$ with $\mathbb{C}(-1)$ such that $H^0(p_{kl})(-1)_{\mathbb{Z}}$ becomes $\frac{1}{2\pi i}\mathbb{Z} \subset \mathbb{C}$ and $Gr_F^1\mathbb{C} = \mathbb{C}$. Then define the map α by

$$\alpha\left([\sum_{i\geq 0}\omega_i + \sum_{[k<l]}\frac{1}{2\pi i}\rho_{kl}\,\Theta]\right) := \sum_{[k<l]}\frac{1}{2\pi i}\rho_{kl}.$$

If $[\sum_{i\geq 0}\omega_i + \sum_{[k<l]}\frac{1}{2\pi i}\rho_{kl}\,\Theta]$ represents an element in $H^1(A^\bullet)_{\mathbb{Z}}$ then it is mapped to $H^0(p_{kl})(-1)_{\mathbb{Z}}$ under α, since all the numbers ρ_{kl} have to be integers (they are the values of an integral along a path around a double point).

Now define the map β in the following way. Given a $\sum_{[k<l]}\frac{1}{2\pi i}\rho_{kl} \in \bigoplus_{[k<l]} H^0(p_{kl})(-1)$, choose a $\sum_{i>0}\omega_i + \sum_{[k<l]}\frac{1}{2\pi i}\rho_{kl}\,\Theta \in A^2$ such that all the $d\omega_i$ have no singularities[3], i. e. $d\omega_i \in E^2(D_i)$. Then we define:

$$\beta\left(\sum_{[k<l]}\frac{1}{2\pi i}\rho_{kl}\right) := \sum_{i>0}[d\omega_i].$$

[3]This is always possible. It can be done similarly to the following: define a bump-function $b : \mathbb{C} \to [0;1]$ around $0 \in \mathbb{C}$ such that $b(z) \equiv 1$ in a small neighbourhood of $0 \in \mathbb{C}$. Then $b(z)\rho\frac{dz}{z}$ has residue ρ and $d\left(b(z)\rho\frac{dz}{z}\right) \in E^2(\mathbb{C})$.

Note that α and β are well-defined and that they yield a short exact sequence (i), where all the maps are strict with respect to the filtrations $F^\bullet$.
ad (ii): The isomorphism sends $[\sum_{i\geq 0}\omega_i] \in W_1H^1(A^\bullet)$ to $\sum_{i\geq 0}[\omega_i] \in \bigoplus_{i>0} H^1(D_i)$. All the remaining verifications are obvious.
ad (iii): The maps γ and δ are defined in the following way:

$$\begin{array}{ccccccc} \bigoplus_{i\geq 0} H^0(D_i) & \xrightarrow{\gamma} & \bigoplus_{[k<l]} H^0(p_{kl}) & \xrightarrow{\delta} & Gr_0^W H^1(A^\bullet) & \to 0 \\ \sum_{i\geq 0} g_i & \mapsto & \sum_{[k<l]}(g_l - g_k) & & & \\ & & \sum_{[k<l]} c_{kl} & \mapsto & [\sum_{[k<l]} c_{kl}\, d\xi] & \end{array}$$

Also this sequence is exact, defined over $\mathbb{Q}$ and respects the filtrations $F^\bullet$. This accomplishes the proof. $\square$

Remark 6.13 *Using the short exact sequences (i), (ii) and (iii) from the proof of 6.11 we may derive that, if $g(D_i)$ denotes the genus of D_i ($g(D_0) = 0$):*

$$\begin{aligned} \dim_{\mathbb{C}} Gr_2^W H^1(A^\bullet) &= (\sum_{[k<l]} 1) - (\sum_{i>0} 1), \\ \dim_{\mathbb{C}} Gr_1^W H^1(A^\bullet) &= \sum_{i\geq 0} 2g(D_i), \\ \dim_{\mathbb{C}} Gr_0^W H^1(A^\bullet) &= (\sum_{[0<k<l]} 1) - (\sum_{i\geq 0} 1) + 1, \end{aligned}$$

which reconfirms the Euler characteristic, that we computed in the proof of Theorem 6.2.

6.4 The Variation of Mixed Hodge Structures is a Nilpotent Orbit

Now it is not difficult anymore to investigate the monodromy of the local system of nearby fundamental groups and to show that the variation of Hodge structures

$$\left(J_{\vec{v},\vec{w}}/J^2_{\vec{v},\vec{w}}\right)^*_{\vec{v}\in(T_0\Delta)^*}$$

is a nilpotent orbit of Hodge structures.

Remark 6.14 The term *nilpotent orbit of mixed Hodge structures* is a generalization (for one variable) of the definition of a nilpotent orbit of MHS given in [Sch73] and [CK82] (Definition 3.1); specifically, [CK82]: (3.1) (iv) is not satisfied. A nilpotent orbit of MHS will be defined properly in Proposition & Definition 6.17.

First of all, we define a nilpotent chain morphism

$$N : A^\bullet \to A^\bullet$$

by defining it in the following way on $B^\bullet$.

Let N be the zero map on $\bigwedge^\bullet(\frac{dp}{p}) \oplus \bigoplus_{i>0} E^\bullet(D_i \log P_i)$ and define it on $E^\bullet(\Delta^1) \otimes \Lambda^\bullet$ as

$$\begin{array}{cccc} N: & E^\bullet(\Delta^1) \otimes \Lambda^\bullet & \to & E^\bullet(\Delta^1) \otimes \Lambda^\bullet \\ & \Xi & \mapsto & \frac{d}{d(-u)}\Xi = -\frac{d}{du}\Xi\,. \end{array}$$

Note that N respects the compatibility relations and hence defines a chain map $N : A^\bullet \to A^\bullet$. Concerning the filtrations, N induces maps

$$N : W_{l+1}A^\bullet \to W_{l-1}A^\bullet \quad \text{and} \quad N : F^p A^\bullet \to F^{p-1}A^\bullet.$$

In particular this chain map induces a map $N : H^1(A^\bullet) \to H^1(A^\bullet)$ (note that we are committing 'abuse of notation' here).

Proposition 6.15 *The filtration $W_\bullet$ on $H^1(A^\bullet)$ and $N : H^1(A^\bullet) \to H^1(A^\bullet)$ satisfy $N^2 = 0$ and*

(i) $W_{-1}H^1(A^\bullet) = 0$

(ii) $W_0\ H^1(A^\bullet) = \operatorname{im}\{N : H^1(A^\bullet) \to H^1(A^\bullet)\}$

(iii) $W_1\ H^1(A^\bullet) = \ker\{N : H^1(A^\bullet) \to H^1(A^\bullet)\}$

(iv) $W_2\ H^1(A^\bullet) = H^1(A^\bullet)$.

Proof: First, recall that if we are given a

$$[\varphi] = [\sum_{i\geq 0} \omega_i + \sum_{[k<l]} K_{kl}\frac{dx}{x} + L_{kl}\frac{dy}{y} + H_{kl}d\xi] \in H^1(A^\bullet),$$

then we can always write

$$[\varphi] = [\sum_{i\geq 0} \omega_i + \sum_{[k<l]} \frac{1}{2\pi i}\rho_{kl}\,\Theta + c_{kl}d\xi],$$

where $\Theta := (1-\xi)\frac{dx}{x} - \xi\frac{dy}{y} - \log t\, d\xi \in W_1\,(E^\bullet \otimes \Lambda^\bullet)^1$. This shows $N^2 = 0$ and $W_{-1}H^1(A^\bullet) = 0$ as well as $W_2\ H^1(A^\bullet) = H^1(A^\bullet)$.

ad(ii): Since the sum of the residues for any closed form ω_i is zero, we have

$$W_0\ H^1(A^\bullet) = \left\{ [\sum_{[k<l]} c_{kl}d\xi] \right\} \text{ and}$$

$$\operatorname{im} N = \left\{ [\sum_{[k<l]} \frac{1}{2\pi i}\rho_{kl}d\xi] \middle| \sum_{[i<l]} \frac{1}{2\pi i}\rho_{il} - \sum_{[k<i]} \frac{1}{2\pi i}\rho_{ki} = 0 \text{ for all } i \right\}.$$

Obviously $\operatorname{im}\{N : H^1(A^\bullet) \to H^1(A^\bullet)\} \subseteq W_0\, H^1(A^\bullet)$. Now let $[\sum_{[k<l]} c_{kl} d\xi]$ be in $W_0\, H^1(A^\bullet)$. Then let $m := \sum_{[k<l]} 1$ and define complex numbers

$$g_\imath := \frac{1}{m}\left(\sum_{[k<\imath]} c_{k\imath} - \sum_{[\imath<l]} c_{\imath l}\right).$$

Note that $\sum_{\imath\geq 0} g_\imath = 0$. It is easy to check now that

$$[\sum_{[k<l]} c_{kl} d\xi] = [\sum_{[k<l]} (c_{kl} - (g_l - g_k))d\xi] \in \operatorname{im} N.$$

ad(iii): Observe that on the one hand

$$W_1\, H^1(A^\bullet) = \left\{[\sum_{\imath>0} \omega_\imath + \sum_{[k<l]} c_{kl} d\xi]\right\}$$

and on the other hand

$$\ker N = \left\{[\sum_{\imath\geq 0} \omega_\imath + \sum_{[k<l]} \frac{1}{2\pi i}\rho_{kl}\,\Theta + c_{kl} d\xi] \,\middle|\, [\sum_{[k<l]} \frac{1}{2\pi i}\rho_{kl}\, d\xi] = 0\right\}.$$

Hence, we have to show: $[\sum_{[k<l]} \frac{1}{2\pi\imath}\rho_{kl}\ d\xi] = 0 \in H^1(A^\bullet) \Rightarrow$ all $\rho_{kl} = 0$. For simplicity define $R_{kl} := \frac{1}{2\pi\imath}\rho_{kl}$. Let $g_\imath \in \mathbb{C}$ be complex numbers such that $d\left(\sum_{\imath\geq 0} g_\imath + \sum_{[k<l]} g_k + \xi(g_l - g_k)\right) = \sum_{[k<l]} R_{kl}\, d\xi$, i e. $(g_l - g_k) = R_{kl}$ $\forall [k<l]$. We show that the real parts, $\Re(R_{kl})$, of all R_{kl} are zero and leave it to the reader to show that the imaginary parts are zero too.

Let i_0 be an index such that $\Re(g_{\imath_0}) = \max_\imath \Re(g_\imath)$. As a consequence, all the real parts of residues at $\omega_{\imath_0}$, that is the numbers $\Re(\operatorname{Res}_{p_{kl}} \omega_{\imath_0})$ for $\{k,l\} \ni i_0$ and $[k<l]$, are non negative ($\operatorname{Res}_{p_{\imath_0 l}} \omega_{\imath_0} = R_{\imath_0 l} = g_{\imath_0} - g_l$ and $\operatorname{Res}_{p_{k\imath_0}} \omega_{\imath_0} = -R_{k\imath_0} = g_{\imath_0} - g_k$). Now the sum of the residues of the form $\omega_{\imath_0}$ has to be zero. This can only happen if for all the indices i with $D_\imath \cap D_{\imath_0} \neq \emptyset$ holds: $g_\imath = g_{\imath_0}$. Since the system od curves $D_\imath$ is connected, this shows $\Re(g_\imath) = \Re(g_{\imath_0})$ for all i and hence $\Re(R_{kl}) = 0$ for all $[k<l]$. □

Remark 6.16 Proposition 6.15 shows that $W_\bullet$ is *the weight filtration of N relative to $W_\bullet$* as defined in [SZ85] (2.5). The proof of Proposition 6.15 is trivial, when one uses Proposition (2.14) in [SZ85].

Let $T \in \operatorname{Aut}\left(H^1(A^\bullet)\right)$ be the monodromy of the local system

$$\left\{\left(J_{\vec{v},\vec{w}}/J^2_{\vec{v},\vec{w}}\right)^*_{\mathbf{Z}}\right\}_{\vec{v}\in(T_0\Delta)^*}.$$

Recall that by means of Theorem 5.8 the automorphism T can be identified with the monodromy of the local system $\{H^1(Z_t;\mathbb{Z})\}_{t\in\Delta^*}$.

A simple example of *a variation of mixed Hodge structure* is given by the following definition. For a complex number $\lambda \in \mathbb{C}^*$ and a nilpotent vectorspace endomorphism A define the endomorphism λ^A to be $e^{\log \lambda A} = \mathbf{1} + \log \lambda A + \frac{1}{2}\log^2 \lambda A^2 + \cdots$.

Proposition & Definition 6.17 Let H be a $\mathbb{Z}$-MHS and $N : H \to H$ be a nilpotent endomorphism such that $N(F^pH) \subset F^{p-1}H$ and $N(W_{l+1}H) \subset W_{l-1}H$. Then for any $\lambda \in \mathbb{C}^*$, the triple

$$H_\lambda := \left(\lambda^{-N} H_{\mathbf{Z}}, (H_{\mathbf{Q}}, W_\bullet), (H_{\mathbf{C}}, W_\bullet, F^\bullet)\right)$$

is a MHS and the family of MHSs $\{H_\lambda\}_{\lambda \in \mathbf{C}^*}$ is called *nilpotent orbit of mixed Hodge structure.* □

In the next theorem we describe, how the lattice $(J_{\vec{v},\vec{w}}/J^2_{\vec{v},\vec{w}})^*_{\mathbf{Z}} = H^1(A^\bullet)_{\mathbf{Z}} \subset H^1(A^\bullet)$ moves, when we move $\vec{v}$ and $\vec{w}$.

Theorem 6.18 *$e^{-2\pi i\, N} = T$ and for λ, $\mu \in \mathbb{C}$ holds:*

$$\left(J_{\lambda\vec{v},\mu\vec{w}}/J^2_{\lambda\vec{v},\mu\vec{w}}\right)^*_{\mathbf{Z}} = \lambda^{-N}\left(J_{\vec{v},\vec{w}}/J^2_{\vec{v},\vec{w}}\right)^*_{\mathbf{Z}} \subset H^1(A^\bullet).$$

Proof: Note that $N^2 = 0$ implies $\lambda^N = \mathbf{1} + \log \lambda N$. Let $\eta = \lambda \times \mu : [0;1] \to (\mathbb{C} \setminus \mathbb{R}^{\leq 0}) \times (\mathbb{C} \setminus \mathbb{R}^{\leq 0})$ be a path with $\eta(0) = (1,1)$ and let $H : [0;1] \times [0;1] \to Z_0$ be a homotopy such that $H(\cdot, s)$ is a path over $\lambda(s)\vec{v}$ based at $\mu(s)\vec{w}$. Define $\gamma_{\vec{v},\vec{w}} := H(\cdot, 0)$.

Using the definitions (a), (b), (c) and (d) on page 94, it is easy to compute for a closed $\varphi \in A^1$:

$$\int_{H(\cdot,s)} \varphi = \int_{\gamma_{\vec{v},\vec{w}}} \varphi + \log \lambda(s) \int_{\gamma_{\vec{v},\vec{w}}} N\varphi = \int_{\gamma_{\vec{v},\vec{w}}} \lambda(s)^N \varphi.$$

Suppose $\int_{\gamma_{\vec{v},\vec{w}}} \varphi$ is an integer, then $\int_{H(\cdot,s)} \lambda(s)^{-N}\varphi$ is an integer too. □

The following statement is a direct consequence of Theorem 6.15 and Theorem 6.18.

Theorem 6.19 *The family of mixed Hodge structures*

$$\left\{\left(J_{\vec{v},\vec{w}}/J^2_{\vec{v},\vec{w}}\right)^*\right\}_{\vec{v} \in (T_0\Delta)^*}$$

is a nilpotent orbit of mixed Hodge structure. □

Consider the graph Γ, whose vertices are the components D_i, where two vertices D_k and D_l are connected by an edge if and only if $D_k \cap D_l \neq \emptyset$. Then we have the following corollary of Theorem 6.15 and Theorem 6.18.

Corollary 6.20 *Γ is a tree if and only if $T = id$.*

Proof: By Theorem 6.15 and Theorem 6.18 $T = id$ is equivalent to $N = 0$ or $\ker N = W_1 H^1(A^\bullet) = W_2 H^1(A^\bullet) = H^1(A^\bullet)$. We observed in Remark 6.13 that holds:

$$\dim_{\mathbb{C}} Gr_2^W H^1(A^\bullet) = (\sum_{[k<l]} 1) - (\sum_{i>0} 1).$$

Since Γ is connected, we have: Γ is a tree $\Leftrightarrow$ #edges = #vertices $- 1$. $\square$

Chapter 7

Iterated Integrals on the Nearby Fundamental Group

Let $I = \sum_{|J|\leq s} a_J \; \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1$ be Chen-closed. We want to define *the iterated integral of this Chen-closed element along a path over $\vec{v}$ (based at $\vec{w}$).*

First, for a sufficiently small $\varepsilon > 0$ and for an arbitrary $\varphi_1 \otimes \cdots \otimes \varphi_r \in \bigotimes^r A^1$, we will define what we call the ε-iterated integral of $\varphi_1 \otimes \cdots \otimes \varphi_r$ along a path γ over $\vec{v}$ (based at $\vec{w}$) and will denote it by:

$$\overset{\varepsilon}{\int_\gamma} \varphi_1 \cdots \varphi_r.$$

This definition depends on the choice of coordinates around the double points (and on ε). But subsequently we will prove that for a *Chen-closed element I* like above and for a path γ over $\vec{v}$ based at $\vec{w}$ the limit

$$\lim_{\varepsilon\to 0} \sum_{|J|\leq s} a_J \overset{\varepsilon}{\int_\gamma} \varphi_{j_1} \cdots \varphi_{j_r} = \lim_{\varepsilon \to 0} \overset{\varepsilon}{\int_\gamma} I$$

exists and depends neither on the choice of coordinates around the double points nor on the choice of γ within a nearby homotopy class. The main tool for the proofs of these assertions is a dga $A^\bullet_{kl}$, which we consider to be $A^\bullet$ localized in a sector at a double point p_{kl}.

These iterated integrals along paths over $\vec{v}$ based at $\vec{w}$ will give a way to describe $\mathrm{Hom}_{\mathbb{Z}}(\vec{J}/\vec{J}^{s+1}, \mathbb{C})$ in terms of the dga $A^\bullet$, similar to Chen's theorem in chapter 1. Finally, this leads to the definition of a MHS on $(\vec{J}/\vec{J}^{s+1})^*$ such that the short exact sequence dual to 6.1 becomes a short exact sequence of MHSs.

7.1 Definition of Iterated Integrals on the Nearby Fundamental Group

Let us begin with the definition of an ε-iterated integral of a $\varphi_1 \otimes \cdots \otimes \varphi_r \in \bigotimes^r A^1$ along a path over $\vec{v}$ (based at $\vec{w}$). Each of the $\varphi_\nu \in A^1$ can be written as

$$\varphi_\nu = \sum_{i\geq 0} \omega_i^{(\nu)} + \sum_{[k<l]} K_{kl}^{(\nu)} \frac{dx}{x} + L_{kl}^{(\nu)} \frac{dy}{y} + H_{kl}^{(\nu)} d\xi.$$

Choose around every double point p_{kl} coordinates $(x,y) : W_{kl} = U_{kl}^k \times U_{kl}^l \to \mathbb{C}^2$ and a coordinate $t : \Delta \to \mathbb{C}$ such that $h_{|W_{kl}}(x,y) = x \cdot y$ and $\vec{v} = \frac{\partial}{\partial t}$. For the double point $p_0 \in D_0$ we require moreover that the coordinate x (which we also called p) satisfies $\frac{\partial}{\partial x} = \frac{\partial}{\partial p} = \vec{w}$. Always, when we talk about ε-iterated integrals in this chapter, we will refer to this coordinate system.

As for the definition of line integrals in 6.2 we give the definition of iterated integrals in several steps.

(a) Let $\gamma : [a;b] \to Z_0$ be a path over $\vec{v}$, which meets the set of double points only once with parameter value $\tau_0 \in]a;b[$, where it changes from D_k to D_l. Define $\gamma_x(\tau)$ (resp. $\gamma_y(\tau)$) to be $x(\gamma(\tau))$ (resp. $y(\gamma(\tau))$) for all $\tau \in [a;b]$ with $\gamma(\tau) \in U_{kl}^k$ (resp. $\gamma(\tau) \in U_{kl}^l$). To shorten the following formulas we abbreviate for $\varepsilon > 0$ small enough:

$$\eta_\varepsilon^{(\nu)} := H_{kl}^{(\nu)}\left(\xi, \log\left(\gamma_x(\tau_0 - \varepsilon) \cdot \gamma_y(\tau_0 + \varepsilon)\right)\right) d\xi.$$

Then define [1]

$$\varepsilon\!\!\int_\gamma \varphi_1 \cdots \varphi_r := \sum_{0\leq\alpha\leq\beta\leq r} \int_{\gamma\leq\tau_0-\varepsilon} \omega_k^{(1)} \cdots \omega_k^{(\alpha)} \int_{[0;1]} \eta_\varepsilon^{(\alpha+1)} \cdots \eta_\varepsilon^{(\beta)} \int_{\gamma\geq\tau_0+\varepsilon} \omega_l^{(\beta+1)} \cdots \omega_l^{(r)}.$$

(b) Let $\gamma : [a;b] \to Z_0$ be a path over $\vec{v}$, which meets the set of double points only once with parameter value $\tau_0 \in]a;b[$, where it stays in one component D_k. Then define:

$$\varepsilon\!\!\int_\gamma \varphi_1 \cdots \varphi_r := \sum_{0\leq\alpha\leq r} \int_{\gamma\leq\tau_0-\varepsilon} \omega_k^{(1)} \cdots \omega_k^{(\alpha)} \int_{\gamma\geq\tau_0+\varepsilon} \omega_k^{(\alpha+1)} \cdots \omega_k^{(r)}.$$

(c) Now let $\gamma : [a;b] \to Z_0$ be a path over $\vec{v}$ such that $\gamma(a)$ and $\gamma(b)$ are no double points. Then there is a finite number of parameter values $\tau_1, \ldots, \tau_N$ with $a < \tau_1 < \cdots < \tau_N < b$, which are mapped onto double points.

[1] Here and also later we make the convention that all the expressions with an integral sign, where from the upper indices the right one is smaller than the left one, like $\int_{[0;1]} \eta_\varepsilon^{(5)} \cdots \eta_\varepsilon^{(4)}$, take the value 1.

Choose a $\tau_i^* \in]\tau_i; \tau_{i+1}[$ for $i = 1, \dots, N-1$ and let $\tau_0^* := a$ and $\tau_N^* := b$. Then define for $i = 1, \dots, N$ the paths $\gamma_i := \gamma|_{[\tau_{i-1}^*, \tau_i^*]}$ with which we define $\varepsilon\!\!\int_\gamma \varphi_1 \cdots \varphi_r$ to be

$$\sum_{0 \le \alpha_1 \le \ \le \alpha_{N-1} \le r} \varepsilon\!\!\int_{\gamma_1} \varphi_1 \cdots \varphi_{\alpha_1} \varepsilon\!\!\int_{\gamma_2} \varphi_{\alpha_1+1} \cdots \varphi_{\alpha_2} \cdots \varepsilon\!\!\int_{\gamma_N} \varphi_{\alpha_{N-1}+1} \cdots \varphi_{\alpha_r}.$$

(d) Finally, let $\gamma : [a; b] \to Z_0$ be a path in Z_0 over $\vec{v}$ based at $\vec{w}$. The coordinate $p : D_0 \to \mathbb{C}$ allows us to consider D_0 as part of the complex plane. There, in the complex plane, we have the differential form (and not the symbol) $\frac{dp}{p}$. Let like before σ be the straight path $\tau \mapsto (1 - \tau)$ in $\mathbb{C}$, then we define

$$\varepsilon\!\!\int_\gamma \varphi_1 \cdots \varphi_r := \varepsilon\!\!\int_{\sigma \star \gamma \star \sigma^{-1}} \varphi_1 \cdots \varphi_r.$$

We leave it to the reader to prove the following proposition. It is a consequence of the corresponding formula for iterated integrals on manifolds (see 1.3 on page 16).

Proposition 7.1 *The definition (c) does not depend on the choice of the τ_i^*. Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r} \in Z_0 \setminus \bigcup_{[k<l]} \{p_{kl}\}$ and let α, $\beta : [0; 1] \to Z_0$ be paths over $\vec{v}$, where $\alpha(0) = \mathbf{p}$, $\alpha(1) = \beta(0) = \mathbf{q}$ and $\beta(1) = \mathbf{r}$. Then for forms $\varphi_1, \dots, \varphi_r \in A^1$ and $\varepsilon > 0$ small enough holds:*

$$\varepsilon\!\!\int_{\alpha \star \beta} = \sum_{0 \le m \le r} \varepsilon\!\!\int_\alpha \varphi_1 \cdots \varphi_m \, \varepsilon\!\!\int_\beta \varphi_{m+1} \cdots \varphi_r$$

□

It is a consequence of the main theorem in the next section (Theorem 7.3) that the following definition of iterated integrals of Chen-closed elements along paths over $\vec{v}$ makes sense.

Definition 7.2 Let $I = \sum_{|J| \le s} a_J \, \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1$ be a Chen-closed element. Then *the iterated integral of I along a path γ over $\vec{v}$ based at $\vec{w}$* is defined by:

$$\int_\gamma \sum_J a_J \, \varphi_{j_1} \cdots \varphi_{j_r} := \lim_{\varepsilon \to 0} \left\{ \varepsilon\!\!\int_\gamma \sum_J a_J \, \varphi_{j_1} \cdots \varphi_{j_r} \right\}.$$

7.2 The Iterated Integrals are Well-Defined

In this section we prove the main theorem of this chapter, which is the following.

Theorem 7.3 *Let $I = \sum_{|J|\leq s} a_J\ \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1$ be a Chen-closed element and let $\gamma : [a;b] \to Z_0$ be a path over $\vec{v}$ based at $\vec{w}$. Then*

$$\int_\gamma \sum_J a_J\ \varphi_{j_1} \cdots \varphi_{j_r}$$

converges and does not depend on the choice of coordinates. Moreover, it does not depend upon the choice of γ within a nearby homotopy class of paths over $\vec{v}$ based at $\vec{w}$, i. e. we have a nearby homotopy functional:

$$\int \sum_J a_J\ \varphi_{j_1} \cdots \varphi_{j_r} : \pi_1(Z_{\vec{v}}, \vec{w}) \to \mathbb{C}.$$

In chapter 1 we saw that one can express iterated integrals over Chen-closed linear combinations of tensor products of smooth 1-forms on a manifiold in different ways as line integrals: One can pull back the iterated integral to the parameter interval according to the definition of iterated integrals. Or one can express these iterated integrals as line integrals on the universal covering. And finally, one can also split up the iterated integral into pieces over parts of the path, which lie in simply connected open sets. Locally – that means over a path lying in such a simply connected open set – iterated integrals can be expressed as line integrals over 1-forms that depend just on the starting- and end point of the path. This is possible because the complex of differential forms in such a local situation is acyclic (cf. 1.14).

The idea to prove 7.3 is to express iterated integrals along paths over $\vec{v}$ in a similar way locally as line integrals along paths over $\vec{v}$. Away from the double points in Z_0 this can be done like described above. But around the double points the problem arises that even if we restrict our complex $A^\bullet$ in the naive way to a complex living on an open neighbourhood of a double point, still the result will not be an acyclic complex. This is due to the fact that $A^\bullet$ restricted to a double point computes the cohomology of the vanishing cycle at that double point.

Since a path over $\vec{v}$ locally at a double point is forced to approach the double point in a particular sector, we can find simply connected open sets in Z_0, which contain all of this (local) path apart from the double point itself. If we are given a closed form φ in A^1 then we can find primitives of this form at least on those open sets. But since φ may have simple poles, these primitives can have logarithmic singularities at the double point.

Recall that one can reduce the length of an iterated integral over a Chen-closed element with differential forms in an acyclic complex. The way to do this is to choose a primitive of a closed form and multiply this function in an appropriate way with the other forms (cf. 1.3). When one takes an iterated integral over a composition of paths, one can take primitives of this closed form over each of the paths and has to take care that their values on the junction point match.

For instance in the case of a path, which passes only through one double point p_{kl} and changes from one component to another, we would like to decompose this path into three parts: two parts, where it stays in one of the components, and a third part, which is represented by the 1-simplex Δ^1. (Observe that also in the definition (a) on page 94 we thought of Δ^1 as being part of the path γ.) But, when we now choose primitives on each of the components of a closed compatible 1-form, the problem arises that these primitives in general have logarithmic singularities right at the junction points of these three parts of the path. Hence, we cannot make their values match. It will turn out that even, if we cannot make their values match, we can make their formal shapes or the type of their singularity match. This leads to a kind of *higher compatibility* and finally to the definition of the dga $A^\bullet_{kl}$.

In this way we can then indeed express locally an iterated integral along a path γ over $\vec{v}$ as line integral of a closed form in A^1_{kl} along γ. This construction allows us to prove the convergence, the independence of local coordinates around the double points and the invariance under nearby homotopies between paths over $\vec{v}$ of iterated integrals of Chen-closed elements.

A crucial role in all convergence considerations is played by the following classical fact (cf. [Cou34] III.2, p. 192).

Fact 7.4 *For $\varepsilon_0 > 0$ let $h :]-\varepsilon_0, \varepsilon_0[\to \mathbb{R}$ be a smooth function that vanishes at 0. Then for any $m \in \mathbb{N}$ holds:*

$$\lim_{\varepsilon \searrow 0} h(\varepsilon) \log^m \varepsilon = 0.$$

7.2.1 A Localization of A^1 in a Sector at a Double Point

Consider a double point p_{kl} in Z_0 with $[k < l]$. Recall that we are given coordinates $(x, y) : W_{kl} = U^k_{kl} \times U^l_{kl} \to \mathbb{C}^2$ around this double point. We want to construct a dga $A^\bullet_{kl}$ for this double point p_{kl}, for which we use these coordinates.

Define $U^{k-}_{kl} := U^k_{kl} \setminus \mathbb{R}^{\leq 0}$ and $U^{l-}_{kl} := U^l_{kl} \setminus \mathbb{R}^{\leq 0}$. Note that the function $\log x$ (resp. $\log y$) can be defined univalently on U^{k-}_{kl} (resp. U^{l-}_{kl}). Now let $E^\bullet(U^{k-}_{kl} \log p_{kl})$ (resp. $E^\bullet(U^{l-}_{kl} \log p_{kl})$) be the sub-dga of $E^\bullet(U^{k-}_{kl})$ (resp. $E^\bullet(U^{l-}_{kl})$), which is generated as $E^\bullet(U^k_{kl})$ (resp. $E^\bullet(U^l_{kl})$)-modules by powers of $\log x$ (resp. $\log y$) and their derivatives. That is, we have embeddings of dga's

$$E^\bullet(U^k_{kl}) \otimes_{\mathbb{C}} \bigwedge\nolimits^\bullet \left(\frac{dx}{x}\right)[\log x] \hookrightarrow E^\bullet(U^{k-}_{kl})$$

and

$$E^\bullet(U^l_{kl}) \otimes_{\mathbb{C}} \bigwedge\nolimits^\bullet \left(\frac{dy}{y}\right)[\log y] \hookrightarrow E^\bullet(U^{l-}_{kl}),$$

whose images are $E^\bullet(U^{k-}_{kl} \log p_{kl})$ and $E^\bullet(U^{l-}_{kl} \log p_{kl})$. [2]

[2] These maps are embeddings for the following reason: If $\sum_{\nu=0}^m g_\nu(x) \log^\nu x \equiv 0$ with $g_\nu \in E^0(U^k_{kl})$ it follows that all $g_\nu \equiv 0$ for $\nu = 0, \ldots, m$, because for any $x_0 \in U^k_{kl}$ the polynomial $\sum_{\nu=0}^m g_\nu(x_0) w^\nu \in \mathbb{C}[w]$ has infinitely many zeroes: $\{\log x_0 + 2\pi i n | n \in \mathbb{Z}\}$.

Since $E^\bullet(U^k_{kl})$ (resp. $E^\bullet(U^l_{kl})$) and $\bigwedge^\bullet(\frac{dx}{x})[\log x]$ (resp. $\bigwedge^\bullet(\frac{dy}{y})[\log y]$) have cohomology only in degree 0, the Künneth formula tells us that also $E^\bullet(U^{k-}_{kl}\log p_{kl})$ (resp. $E^\bullet(U^{l-}_{kl}\log p_{kl})$) have no cohomology except in degree 0.

If $\{k, l\} = \{0, 1\}$ then define

$$E^\bullet(U^{0-}_{01}\log p_{01}) := \bigwedge{}^\bullet(\frac{dp}{p})[\log p],$$

which are the polynomials in the variable $\log p$ and coefficients in $\bigwedge^\bullet(\frac{dp}{p})$, where the differential is given by the Leibniz rule, $d\,\frac{dp}{p} = 0$ and $d\,\log p = \frac{dp}{p}$.

Now let $\Lambda^\bullet_{kl} := \bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})[v, w]$ be the dga, which is defined as follows: Like before let $\bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})$ be the free graded-commutative $\mathbb{C}$-algebra with unit generated by the symbols $\frac{dx}{x}$, $\frac{dy}{y}$ in degree 1.

$\Lambda^\bullet := \bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})[v, w]$ is then the dga $\bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})[v, w]$ of polynomials in two variables v and w with coefficients in $\bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})$, where the differential d is given by the Leibniz rule, $d\left(\frac{dx}{x}\right) = d\left(\frac{dy}{y}\right) = 0$ and

$$dv := \frac{dx}{x} \qquad \text{and} \qquad dw := \frac{dy}{y}.$$

Observe that there is an embedding

$$\bigwedge{}^\bullet(\frac{dx}{x}, \frac{dy}{y})[u] \hookrightarrow \bigwedge{}^\bullet(\frac{dx}{x}, \frac{dy}{y})[v, w]$$

by setting $u = v + w$. Note that $(\Lambda^\bullet_{kl}, d)$ only has cohomology in degree 0.

Similar like in the definition of $A^\bullet$ we define the dga $A^\bullet_{kl}$ as sub-dga of

$$B^\bullet_{kl} := E^\bullet(U^{k-}_{kl}\log p_{kl}) \oplus E^\bullet(U^{l-}_{kl}\log p_{kl}) \oplus E^\bullet(\Delta^1) \otimes_{\mathbb{C}} \Lambda^\bullet_{kl},$$

which contains all those elements that are subject to the following compatibility conditions, which generalize the compatibility conditions in the definition of $A^\bullet$. Observe that $B^n_{kl} = 0$ for $n \geq 4$.

We shall use P_{kl}, K_{kl}, L_{kl}, H_{kl}, R_{kl}, S_{kl} and T_{kl} now to denote elements in $\mathbb{C}[\xi, v, w]$. If the context is clear, we omit the indices k, l.

$\mathbf{A^0_{kl}}$: An element

$$f = \sum_{\nu=0}^{m} g_{k,\nu}(x)\log^\nu x + \sum_{\nu=0}^{m} g_{l,\nu}(y)\log^\nu y + P_{kl}(\xi, v, w) \in B^0_{kl}$$

is called *compatible*, iff holds:

$$P_{kl}(0, v, w) = \sum_{\nu=0}^{m} g_{k,\nu}(0)v^\nu \text{ and } P_{kl}(1, v, w) = \sum_{\nu=0}^{m} g_{l,\nu}(0)w^\nu.$$

$\mathbf{A}^1_{\mathbf{kl}}$: We call an element in B^1_{kl},

$$\varphi = \sum_{\nu=0}^{m} \omega_{k,\nu} \log^\nu x + \sum_{\nu=0}^{m} \omega_{l,\nu} \log^\nu y + K \frac{dx}{x} + L \frac{dy}{y} + H\, d\xi$$

a *compatible* element iff it satifies:

$$\begin{aligned} K(0,v,w) &= \textstyle\sum_{\nu=0}^m \mathrm{Res}_{p_{kl}}\, \omega_{k,\nu}\, v^\nu \quad , \quad L(0,v,w) = 0, \\ K(1,v,w) &= \quad 0 \quad , \quad L(1,v,w) = \sum_{\nu=0}^m \mathrm{Res}_{p_{kl}}\, \omega_{l,\nu}\, w^\nu. \end{aligned}$$

$\mathbf{A}^2_{\mathbf{kl}}$: Let us call an element

$$\phi = \Omega^{(k)} + \Omega^{(l)} + R\, d\xi \wedge \frac{dx}{x} + S\, d\xi \wedge \frac{dy}{y} + T \frac{dx}{x} \wedge \frac{dy}{y} \in B^2_{kl}$$

compatible iff holds:

$$T(0,v,w) = T(1,v,w) = 0.$$

$\mathbf{A}^3_{\mathbf{kl}}$: And finally let $A^3_{kl} := B^3_{kl}$.

Note that $dA^i_{kl} \subset A^{i+1}_{kl}$ for $i \geq 0$ and that $A^\bullet_{kl}$ is a dga.

Again, similar like in the definition of $A^\bullet$, we can give an alternative definition of $A^\bullet_{kl}$ as follows. Consider the surjective map of complexes

$$\Phi_{kl} : B^\bullet_{kl} \longrightarrow \Lambda^\bullet_{kl} \oplus \Lambda^\bullet_{kl},$$

which sends $\sum_{\nu=0}^m g_{k,\nu}(x) \log^\nu x + \sum_{\nu=0}^m g_{l,\nu}(y) \log^\nu y + P_{kl}(\xi, v, w) \in B^0_{kl}$ to $(P_{kl}(0,v,w) - \sum_{\nu=0}^m g_{k,\nu}(0)v^\nu) \oplus (P_{kl}(1,v,w) - \sum_{\nu=0}^m g_{l,\nu}(0)w^\nu)$ and

$$\sum_{\nu=0}^{m} \omega_{k,\nu} \log^\nu x + \sum_{\nu=0}^{m} \omega_{l,\nu} \log^\nu y + K(\xi,v,w)\frac{dx}{x} + L(\xi,v,w)\frac{dy}{y} + H(\xi,v,w) d\xi$$

in B^1_{kl} to

$$\left[\left(K(0,v,w) - \sum_{\nu=0}^m \mathrm{Res}_{p_{kl}}\, \omega_{k,\nu}\, v^\nu\right) \frac{dx}{x} + L(0,v,w)\frac{dy}{y}\right] \oplus \left[K(1,v,w)\frac{dx}{x} + \left(L(1,v,w) - \sum_{\nu=0}^m \mathrm{Res}_{p_{kl}}\, \omega_{l,\nu}\, w^\nu\right) \frac{dy}{y}\right].$$

Moreover, Φ_{kl} maps

$$\Omega^{(k)} + \Omega^{(l)} + R(\xi,v,w)\, d\xi \wedge \frac{dx}{x} + S(\xi,v,w)\, d\xi \wedge \frac{dy}{y} + T(\xi,v,w) \frac{dx}{x} \wedge \frac{dy}{y} \in B^2_{kl}$$

to $T(0,v,w)\frac{dx}{x} \wedge \frac{dy}{y} \oplus T(1,v,w)\frac{dx}{x} \wedge \frac{dy}{y}$. It is easy to check that Φ is indeed a surjective map of complexes. Then we find:

$$A^\bullet_{kl} = \ker \Phi_{kl}$$

and with $C^\bullet_{kl} := \Lambda^\bullet_{kl} \oplus \Lambda^\bullet_{kl}$ we get the short exact sequence

$$0 \longrightarrow A^\bullet_{kl} \longrightarrow B^\bullet_{kl} \longrightarrow C^\bullet_{kl} \longrightarrow 0$$

and may conclude from the long exact cohomology sequence using $H^0(A^\bullet_{kl}) = \mathbb{C}$ that $H^i(A^\bullet_{kl}) = 0$ for $i \geq 1$.

Note that there is an obvious dga-morphism:

$$A^\bullet \to A^\bullet_{kl}.$$

7.2.2 A Path Locally at a Double Point

Consider a path over $\vec{v}$, which lies entirely in $U^{k-}_{kl} \cup \{p_{kl}\} \cup U^{l-}_{kl}$ for a double point p_{kl}. We want to extend the definitions (a), (b), (c) and (d) of page 110 to the definition of ε-iterated integrals with forms in A^1_{kl} along this path. Again like before we define these ε-iterated integrals in several steps. Finally for a Chen-closed element we define again the iterated integral of this element along the path as the limit of the sum of ε-iterated integrals.

Let $\varphi_1 \otimes \cdots \otimes \varphi_r$ in $\bigotimes^r A^1_{kl}$, where we write each

$$\varphi_j = \tilde{\omega}^{(j)}_k + \tilde{\omega}^{(j)}_l + \Xi^{(j)}$$

with

$$\tilde{\omega}^{(j)}_k = \sum_{\nu=0}^{m} \omega^{(j)}_{k,\nu} \log^\nu x \qquad \text{and} \qquad \tilde{\omega}^{(j)}_l = \sum_{\nu=0}^{m} \omega^{(j)}_{l,\nu} \log^\nu y$$

and

$$\Xi^{(j)} = K^{(j)}\frac{dx}{x} + L^{(j)}\frac{dy}{y} + H^{(j)}d\xi.$$

We define now the iterated integral of $\varphi_1 \otimes \cdots \otimes \varphi_r$ along a path over $\vec{v}$ in $U^{k-}_{kl} \cup \{p_{kl}\} \cup U^{l-}_{kl}$ as follows.

(a) Let $\gamma : [a;b] \to U^{k-}_{kl} \cup \{p_{kl}\} \cup U^{l-}_{kl}$ be a path over $\vec{v}$, which meets the set of double points only once with parameter value $\tau_0 \in]a;b[$, where it changes from D_k to D_l. Define $\gamma_x(\tau)$ (resp. $\gamma_y(\tau)$) to be $x(\gamma(\tau))$ (resp. $y(\gamma(\tau))$). We abbreviate for $\varepsilon > 0$ small enough:

$$\eta^{(j)}_\varepsilon := H^{(j)}\big(\xi, \log\left(\gamma_x(\tau_0 - \varepsilon)\cdot\gamma_y(\tau_0+\varepsilon)\right)\big)\, d\xi.$$

Then define

$$\overset{\varepsilon}{\oint_\gamma} \varphi_1 \cdots \varphi_r := \sum_{0\leq\alpha\leq\beta\leq r} \int_{\gamma\leq\tau_0-\varepsilon} \tilde{\omega}^{(1)}_k \cdots \tilde{\omega}^{(\alpha)}_k \int_{[0,1]} \eta^{(\alpha+1)}_\varepsilon \cdots \eta^{(\beta)}_\varepsilon \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}^{(\beta+1)}_l \cdots \tilde{\omega}^{(r)}_l.$$

(b) Let $\gamma : [a;b] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ be a path over $\vec{v}$, which meets the set of double points only once with parameter value $\tau_0 \in]a;b[$, where it stays in one component D_k. Then define:

$$\oint_\gamma \varphi_1 \cdots \varphi_r := \sum_{0 \le \alpha \le r} \int_{\gamma \le \tau_0 - \epsilon} \tilde{\omega}_k^{(1)} \cdots \tilde{\omega}_k^{(\alpha)} \int_{\gamma \ge \tau_0 + \epsilon} \tilde{\omega}_k^{(\alpha+1)} \cdots \tilde{\omega}_k^{(r)}.$$

(c) Now let $\gamma : [a;b] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ be a path over $\vec{v}$ such that $\gamma(a)$ and $\gamma(b)$ are different from p_{kl}. Then there is a finite number of parameter values $\tau_1, \ldots, \tau_N$ with $a < \tau_1 < \cdots < \tau_N < b$, which are mapped onto p_{kl}. Choose a $\tau_i^* \in]\tau_i; \tau_{i+1}[$ for $i = 1, \ldots, N-1$ and let $\tau_0^* := a$ and $\tau_N^* := b$. Then we define $\gamma_i := \gamma|_{[\tau_{i-1}^*, \tau_i^*]}$ for $i = 1, \ldots, N$ and let: $\oint_\gamma \varphi_1 \cdots \varphi_r :=$

$$\sum_{0 \le \alpha_1 \le \quad \le \alpha_{N-1} \le r} \oint_{\gamma_1} \varphi_1 \cdots \varphi_{\alpha_1} \oint_{\gamma_2} \varphi_{\alpha_1+1} \cdots \varphi_{\alpha_2} \cdots \oint_{\gamma_N} \varphi_{\alpha_{N-1}+1} \cdots \varphi_{\alpha_r}.$$

(d) Finally assume that $p_{kl} = p_0$. Let $\gamma : [a;b] \to \{p_{01}\} \cup U_{01}^{1-}$ be a path over $\vec{v}$ starting (or ending) at $\vec{w}$. Like before, the coordinate $p : D_0 \to \mathbb{C}$ allows us to consider D_0 as part of the complex plane. There, in the complex plane, we have the differential form (and not the symbol) $\frac{dp}{p}$. Let σ be the straight path $\tau \mapsto (1-\tau)$ in $\mathbb{C}$, then we define $\varepsilon > 0$ small enough

$$\oint_\gamma \varphi_1 \cdots \varphi_r := \oint_{\sigma \star \gamma} \varphi_1 \cdots \varphi_r \quad \left(\text{or } \oint_{\gamma \star \sigma^{-1}} \varphi_1 \cdots \varphi_r \right).$$

So, we can define iterated integrals of Chen-closed elements with forms in this localization $A_{kl}^\bullet$ of $A^\bullet$ in a sector at p_{kl}.

Definition 7.5 Let $I_{kl} = \sum_{|J| \le s} a_J \, \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^s \bigotimes^r A_{kl}^1$ be a Chen-closed element. Then *the iterated integral of I along a path* $\gamma : [a;b] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ *over* $\vec{v}$ is defined by:

$$\int_\gamma \sum_J a_J \, \varphi_{j_1} \cdots \varphi_{j_r} := \lim_{\varepsilon \to 0} \left\{ \oint_\gamma \sum_J a_J \, \varphi_{j_1} \cdots \varphi_{j_r} \right\}.$$

Note that if all the φ_j are even in the image of the obvious dga morphism $A^\bullet \to A_{kl}^\bullet$, then clearly these definitions coincide with the definitions on the pages 110 and 111.

Proposition 7.6 *Let $f \in A_{kl}^0$. Then for any path $\gamma : [a;b] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ over $\vec{v}$ holds:*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

Proof: Let $\varphi = \tilde{\omega}_k + \tilde{\omega}_l + K\frac{dx}{x} + L\frac{dy}{y} + Hd\xi = df = d\left(G_k(x) + G_l(y) + P(\xi, v, w)\right)$ and write

$$\tilde{\omega}_k = \sum_{\nu=0}^{m} \omega_{k,\nu} \log^\nu x \qquad \text{and} \qquad \tilde{\omega}_l = \sum_{\nu=0}^{m} \omega_{l,\nu} \log^\nu y$$

as well as

$$G_k(x) = \sum_{\nu=0}^{m} g_{k,\nu}(x) \log^\nu x \qquad \text{and} \qquad G_l = \sum_{\nu=0}^{m} g_{l,\nu}(y) \log^\nu y.$$

We can decompose γ into paths γ_i,for $i = 1, \ldots, N$, i. e. $\gamma = \gamma_1 \star \cdots \star \gamma_N$ such that each of the paths γ_i meets the double point p_{kl} only once. Because line integrals are additive we have

$$\int_\gamma \varphi = \sum_{i=1}^{N} \int_{\gamma_i} \varphi,$$

provided that all the integrals on the right hand side converge. Therefore we may assume without loss of generality that γ meets the double point p_{kl} only once with parameter value $\tau_0 \in]a; b[$. We distinguish two cases.

A path traversing the double point: Suppose that γ changes components – say, it runs from D_k to D_l. Observe first that holds:

$$\int_{[0;1]} H(\xi, v, w) d\xi = P(1, v, w) - P(0, v, w).$$

Then we may compute – using the compatibility:

$$\begin{aligned}
&\int_{\gamma \leq \tau_0 - \epsilon} dG_k - P(0, \log \gamma_x(\tau_0 - \varepsilon), \log \gamma_y(\tau_0 + \varepsilon)) \\
=&G_k(\gamma_x(\tau_0 - \varepsilon)) - \sum_{\nu=0}^{m} g_{k,\nu}(0) \log^\nu \gamma_x(\tau_0 - \varepsilon) - G_k(\gamma_x(a)) \\
=&\sum_{\nu=0}^{m} \left(g_{k,\nu}(\gamma_x(\tau_0 - \varepsilon)) - g_{k,\nu}(0)\right) \log^\nu \gamma_x(\tau_0 - \varepsilon) - G_k(\gamma_x(a)).
\end{aligned}$$

With 7.4 we derive

$$\lim_{\varepsilon \to 0} \int_{\gamma \leq \tau_0 - \epsilon} \sum_{\nu=0}^{m} \omega_{k,\nu} \log^\nu x - P(0, \log \gamma_x(\tau_0 - \varepsilon), \log \gamma_y(\tau_0 + \varepsilon)) = -G_k(\gamma_x(a))$$

and similarly we find:

$$\lim_{\varepsilon \to 0} \int_{\gamma \geq \tau_0 + \epsilon} \sum_{\nu=0}^{m} \omega_{l,\nu} \log^\nu y - P(1, \log \gamma_x(\tau_0 - \varepsilon), \log \gamma_y(\tau_0 + \varepsilon)) = G_l(\gamma_y(b)).$$

A path colliding with the double point: Suppose now that γ stays in one component D_k. Here we compute:

$$\begin{aligned}&\int\limits_{\gamma\leq\tau_0-\varepsilon}\tilde{\omega}_k+\int\limits_{\gamma\geq\tau_0+\varepsilon}\tilde{\omega}_k\\ =\quad&(G_k(\gamma_x(\tau_0-\varepsilon))-G_k(\gamma_y(\tau_0+\varepsilon)))+(G_k(\gamma_x(b))-G_k(\gamma_x(a)))\end{aligned}$$

and the first summand tends to 0 as ε becomes small. □

Theorem 7.7 *Let p, $q\in U_{kl}^{k-}\cup U_{kl}^{l-}$ and let*

$$I_{kl}=\sum_{|J|\leq s}a_J\,\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_r}\in\bigoplus_{r=1}^{s}\bigotimes^{r}A_{kl}^{1}$$

be a Chen-closed element. Then there is a "function" $f_{pq}\in A_{kl}^0$ such that for any path $\gamma:[a;b]\to Z_0$ over $\vec{v}$, which lies in $U_{kl}^{k-}\cup\{p_{kl}\}\cup U_{kl}^{l-}$ and which starts in p and ends in q, holds:

$$\int_\gamma I_{kl}=f_{pq}(q)-f_{pq}(p).$$

Proposition 7.6 and Theorem 7.7 together yield:

Corollary 7.8 *$\int_\gamma I_{kl}$ converges under the assumptions of Theorem 7.7.*

To perform all the convergence considerations in the proof of Theorem 7.7 we need the following technical but elementary Lemma.

Lemma 7.9 *Let $\gamma:[a;b]\to U_{kl}^{k-}\cup\{p_{kl}\}\cup U_{kl}^{l-}$ be a path over $\vec{v}$ that meets the double point once with parameter value τ_0 and changes from D_k to D_l. Let $\tilde{\omega}_k^{(1)},\ldots,\tilde{\omega}_k^{(r)}\in E^1(U_{kl}^{k-}\log p_{kl})$, let $\tilde{\omega}_l^{(1)},\ldots,\tilde{\omega}_l^{(r)}\in E^1(U_{kl}^{l-}\log p_{kl})$ and let $h^{(1)},\ldots,h^{(r)}\in\mathbb{C}[\xi,v,w]$. Define*

$$\eta_\varepsilon^{(j)}:=h^{(j)}\big(\xi,\log\gamma_x(\tau_0-\varepsilon),\log\gamma_y(\tau_0+\varepsilon)\big)d\xi.$$

Then there is a positive number C and a natural number N such that for all $\varepsilon>0$ small enough holds:

$$\left|\int_{\gamma\leq\tau_0-\varepsilon}\tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(r)}\right|\;\leq\;C\left|\log^N\varepsilon\right|\tag{7.1}$$

$$\left|\int_{[0;1]}\eta_\varepsilon^{(1)}\cdots\eta_\varepsilon^{(r)}\right|\;\leq\;C\left|\log^N\varepsilon\right|\tag{7.2}$$

$$\left|\int_{\gamma\geq\tau_0-\varepsilon}\tilde{\omega}_l^{(1)}\cdots\tilde{\omega}_l^{(r)}\right|\;\leq\;C\left|\log^N\varepsilon\right|.\tag{7.3}$$

Also the corresponding assertion holds for a path γ over $\vec{v}$, which meets the double point once but stays in one component.

Proof: For the proof of (7.2) in Lemma 7.9, observe that

$$\int_{[0;1]} \left(h^{(1)}(\xi, v, w)d\xi\right) \cdots \left(h^{(r)}(\xi, v, w)d\xi\right)$$

is a polynomial, say $Q(v, w)$, in $\mathbb{C}[v, w]$ such that

$$\int_{[0;1]} \eta_\varepsilon^{(1)} \cdots \eta_\varepsilon^{(r)} = Q(\log \gamma_x(\tau_0 - \varepsilon), \log \gamma_y(\tau_0 + \varepsilon)).$$

Write $Q(v, w) = \sum_{m,n} a_{mn} v^m w^n$. Then we find

$$Q(\log \gamma_x(\tau_0 - \varepsilon), \log \gamma_y(\tau_0 + \varepsilon))$$

$$= \sum_{m,n} a_{mn} \left(\log(\frac{\gamma_x(\tau_0 - \varepsilon)}{\varepsilon}) + \log \varepsilon\right)^m \left(\log(\frac{\gamma_y(\tau_0 + \varepsilon)}{\varepsilon}) + \log \varepsilon\right)^n.$$

Assertion (7.2) is true since:

$$\lim_{\varepsilon \to 0} \log \left(\frac{\gamma_x(\tau_0 - \varepsilon)}{\varepsilon}\right) = -\log \dot{\gamma}_x(\tau_0) \quad \text{and} \quad \lim_{\varepsilon \to 0} \log \left(\frac{\gamma_y(\tau_0 + \varepsilon)}{\varepsilon}\right) = \log \dot{\gamma}_y(\tau_0).$$

(7.3) follows from (7.1) by the second part of Proposition 1.3 and hence just (7.1) remains to be shown. Let $f_j(\tau)d\tau = \gamma_x^* \tilde{\omega}_k^{(i)}$ for $i = 1, \ldots, r$. Then

$$\begin{aligned}
&\left| \int_{\gamma \le \tau_0 - \varepsilon} \tilde{\omega}_k^{(1)} \cdots \tilde{\omega}_k^{(r)} \right| \\
\le &\left| \int_{a \le \tau_1 \le \cdots \le \tau_r \le \tau_0 - \varepsilon} \cdots \int f_1(\tau_1) \cdots f_r(\tau_r) d\tau_1 \cdots d\tau_r \right| \\
\le &\int_a^{\tau_0 - \varepsilon} |f_1(\tau_1)| d\tau_1 \cdot \cdots \cdot \int_a^{\tau_0 - \varepsilon} |f_r(\tau_r)| d\tau_r
\end{aligned}$$

and if $\tilde{\omega}_k(j) = \sum_{\nu=0}^m A_\nu(x) \log^\nu x \frac{dx}{x} + B_\nu(x) \log^\nu x d\bar{x}$, then we find

$$f_j(\tau) = \sum_{\nu=0}^m \left\{ \frac{A_\nu(\gamma_x(\tau)) \log^\nu \gamma_x(\tau)}{\gamma_x(\tau)} \dot{\gamma}_x(\tau) + B_\nu(\gamma_x(\tau)) \log^\nu \gamma_x(\tau) \overline{\dot{\gamma}_x(\tau)} \right\}.$$

There are A_ν, $B_\nu > 0$ such that $\sup_x |A_\nu(x)| \le A_\nu$ and $\sup_x |B_\nu(x)| \le b_\nu$.

Moreover let $C_\nu := \int_a^{\tau_0} |\overline{\dot\gamma_x(\tau)}| d\tau$. Then

$$
\begin{aligned}
&\int_a^{\tau_0-\varepsilon} |f_J(\tau)| d\tau \\
\leq\ & \sum_{\nu=0}^{m} \left\{ A_\nu \int_a^{\tau_0-\varepsilon} \left| \frac{\log^\nu \gamma_x(\tau)}{\gamma_x(\tau)} \dot\gamma_x(\tau) \right| d\tau + B_\nu \int_a^{\tau_0-\varepsilon} \left| \log^\nu \gamma_x(\tau) \overline{\dot\gamma_x(\tau)} \right| \right\} \\
\leq\ & \sum_{\nu=0}^{m} \left\{ A_\nu \left| \frac{1}{\nu+1} \left(\log^{\nu+1} \gamma_x(\tau_0-\varepsilon) - \log^{\nu+1} \gamma_x(a) \right) \right| \right. \\
+\ & \left. B_\nu \left| \log^\nu \gamma_x(\tau_0-\varepsilon) \right| C_\nu \right\}
\end{aligned}
$$

The assertion follows now from the differentability of γ_x. □

Proof of 7.7: We will prove the theorem by induction on the maximal length s of tensor products in I_{kl}.

$s = 1$: trivial.

$s \geq 2$: Let $I_{kl} = \sum_J a_J\, \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}$ be Chen-closed. According to Lemma 1.10 there is for each J with $|J| = s$ at least one closed and hence exact form $\varphi_{j_\lambda} = df_{j_\lambda}$.

We distinguish three cases:

(i) If $\lambda = 1$, then define $\tilde I_{j_1, \quad, j_s} := (f_{j_1} - f_{j_1}(p))\, \varphi_{j_2} \otimes \cdots \otimes \varphi_{j_s} \in \bigotimes^{s-1} A^1_{kl}$.

(ii) If $1 < \lambda < s$ define $\tilde I_{j_1, \quad, j_s} \in \bigotimes^{s-1} A^1_{kl}$ to be:

$$\varphi_{j_1} \otimes \cdots \otimes f_{j_\lambda} \varphi_{j_{\lambda+1}} \otimes \cdots \otimes \varphi_{j_s} - \varphi_{j_1} \otimes \cdots \otimes f_{j_\lambda} \varphi_{j_{\lambda-1}} \otimes \cdots \otimes \varphi_{j_s}.$$

(iii) And finally set $\tilde I_{j_1, \quad, j_s} := -\varphi_{j_1} \otimes \cdots \otimes (f_{j_s} - f_{j_s}(q))\, \varphi_{j_{s-1}} \in \bigotimes^{s-1} A^1_{kl}$, if $\lambda = s$.

Note that the definition of the $\tilde I_{j_1, \quad, j_s}$ only depends on p and q.

Now

$$\tilde I_{kl} := \sum_{|J|=s} a_J\, \tilde I_{j_1, \quad, j_s} + \sum_{|J| \leq s-1} a_J\, \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1_{kl}$$

is Chen-closed and by induction hypothesis there is a closed $\varphi_{kl} \in A^1_{kl}$ such that $\int_\gamma \tilde I_{kl} = \int_\gamma \varphi_{pq}$. It remains to show that

$$\int_\gamma I_{kl} = \int_\gamma \tilde I_{kl}. \tag{7.4}$$

We will prove (7.4) by showing that in the cases (i), (ii) and (iii) for each J with $|J| = s$ holds:

$$\lim_{\varepsilon \to 0} \left\{ \sideset{^\varepsilon}{}\int_\gamma \varphi_{j_1} \ldots \varphi_{j_s} - \sideset{^\varepsilon}{}\int_\gamma \tilde I_{j_1, \quad, j_s} \right\} = 0. \tag{7.5}$$

This implies then $\int_\gamma I_{kl} = \int_\gamma \tilde{I}_{kl}$.

Proof of (7.5): For simplicity let $(j_1, \dots, j_s) = (1, \dots, s)$. Write

$$\varphi_j = \tilde{\omega}_k^{(j)} + \tilde{\omega}_l^{(j)} + K^{(j)}\frac{dx}{x} + L^{(j)}\frac{dy}{y} + H^{(j)}d\xi$$

and

$$f_\lambda = f = G_k(x) + G_l(y) + P(\xi, v, w),$$

where

$$\tilde{\omega}_k = \sum_{\nu=0}^{m} \omega_{k,\nu} \log^\nu x \qquad \text{and} \qquad \tilde{\omega}_l = \sum_{\nu=0}^{m} \omega_{l,\nu} \log^\nu y$$

and

$$G_k(x) = \sum_{\nu=0}^{m} g_{k,\nu}(x) \log^\nu x \qquad \text{and} \qquad G_l(y) = \sum_{\nu=0}^{m} g_{l,\nu}(y) \log^\nu y.$$

Before we show 7.5 for general paths γ over $\vec{v}$ in $U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ we again consider two special cases first.

A path traversing the double point once: Suppose that $\gamma : [a; b] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ meets the double point p_{kl} once with parameter value$\tau_0 \in]a; b[$, where it changes from D_k to D_l. Again, we make use of the notation:

$$P_\varepsilon(\xi) := P(\xi, \log \gamma_x(\tau_0 - \varepsilon), \log \gamma_y(\tau_0 + \varepsilon))$$

and

$$\eta_\varepsilon^{(j)} := H^{(j)}(\xi, \log \gamma_x(\tau_0 - \varepsilon), \log \gamma_y(\tau_0 + \varepsilon))d\xi.$$

Note that: $dP_\varepsilon(\xi) = \eta_\varepsilon^{(\lambda)}$ in $E^\bullet(\Delta^1)$.

ad(i) If $\lambda = 1$ we argue as follows. Assume without loss of generality that $f(p) = f(\gamma(a)) = 0$. Then we compute as follows with $\varphi_1 = df$:

$$\begin{aligned}
&\varepsilon\!\!\int_\gamma (f\varphi_2)\varphi_3 \cdots \varphi_s \\
= &\int_{\gamma \geq \tau_0 + \varepsilon} (G_l\tilde{\omega}_l^{(2)})\tilde{\omega}_l^{(3)} \cdots \tilde{\omega}_l^{(s)} \\
&+ \sum_{1=\alpha<\beta\leq s} \int_{[0;1]} (P_\varepsilon \eta_\varepsilon^{(2)})\eta_\varepsilon^{(3)} \cdots \eta_\varepsilon^{(\beta)} \int_{\gamma \geq \tau_0 + \varepsilon} \tilde{\omega}_l^{(\beta+1)} \cdots \tilde{\omega}_l^{(s)} \\
&+ \sum_{1<\alpha\leq\beta\leq s} \int_{\gamma \leq \tau_0 - \varepsilon} (G_k\tilde{\omega}_k^{(2)})\tilde{\omega}_k^{(3)} \cdots \tilde{\omega}_k^{(\alpha)} \int_{[0,1]} \eta_\varepsilon^{(\alpha+1)} \cdots \eta_\varepsilon^{(\beta)} \int_{\gamma \geq \tau_0 + \varepsilon} \tilde{\omega}_l^{(\beta+1)} \cdots \tilde{\omega}_l^{(s)}
\end{aligned}$$

$$= \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(1)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ \sum_{1=\alpha<\beta\leq s} \int_{[0;1]} \eta_\varepsilon^{(1)}\cdots\eta_\varepsilon^{(\beta)} \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(\beta+1)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ \sum_{1<\alpha\leq\beta\leq s} \int_{\gamma\leq\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)} \int_{[0;1]} \eta_\varepsilon^{(\alpha+1)}\cdots\eta_\varepsilon^{(\beta)} \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(\beta+1)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ G_l(\gamma_y(\tau_0+\varepsilon)) \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(2)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ \sum_{1=\alpha<\beta\leq s} P_\varepsilon(0) \int_{[0;1]} \eta_\varepsilon^{(2)})\eta_\varepsilon^{(3)}\cdots\eta_\varepsilon^{(\beta)} \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(\beta+1)}\cdots\tilde{\omega}_l^{(s)}$$

$$= \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(1)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ \sum_{1=\alpha<\beta\leq s} \int_{[0;1]} \eta_\varepsilon^{(1)}\cdots\eta_\varepsilon^{(\beta)} \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(\beta+1)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ \sum_{1<\alpha\leq\beta\leq s} \int_{\gamma\leq\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)} \int_{[0;1]} \eta_\varepsilon^{(\alpha+1)}\cdots\eta_\varepsilon^{(\beta)} \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(\beta+1)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ \int_{[0,1]} \eta_\varepsilon^{(1)} \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(2)}\cdots\tilde{\omega}_l^{(s)} + \int_{\gamma\leq\tau_0-\varepsilon} \tilde{\omega}_k^{(1)} \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(2)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ \sum_{1=\alpha<\beta\leq s} \int_{\gamma\leq\tau_0-\varepsilon} \tilde{\omega}_k^{(1)} \int_{[0;1]} \eta_\varepsilon^{(2)}\cdots\eta_\varepsilon^{(\beta)} \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(\beta+1)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ [(G_l(\gamma_y(\tau_0+\varepsilon)) - P_\varepsilon(1)) + (P_\varepsilon(0) - G_k(\gamma_x(\tau_0-\varepsilon)))] \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(2)}\cdots\tilde{\omega}_l^{(s)}$$
$$+ \sum_{1=\alpha<\beta\leq s} (P_\varepsilon(0) - G_k(\log\gamma_x(\tau_0-\varepsilon)) \int_{[0;1]} \eta_\varepsilon^{(2)}\cdots\eta_\varepsilon^{(\beta)} \int_{\gamma\geq\tau_0+\varepsilon} \tilde{\omega}_l^{(\beta+1)}\cdots\tilde{\omega}_l^{(s)}$$

$$= \varepsilon\!\!\int_\gamma \varphi_1\cdots\varphi_s$$
$$+ [(G_l(\gamma_y(\tau_0+\varepsilon)) - P_\varepsilon(1)) + (P_\varepsilon(0) - G_k(\gamma_x(\tau_0-\varepsilon)))]$$

$$
\begin{aligned}
&\int\limits_{\gamma\geq\tau_0+\epsilon} \tilde{\omega}_l^{(2)}\cdots\tilde{\omega}_l^{(s)} \\
&+ \sum_{1=\alpha<\beta\leq s} (P_\varepsilon(0) - G_k(\log\gamma_x(\tau_0-\varepsilon)) \\
&\cdot \int\limits_{[0;1]} \eta_\varepsilon^{(2)}\cdots\eta_\varepsilon^{(\beta)} \int\limits_{\gamma\geq\tau_0+\epsilon} \tilde{\omega}_l^{(\beta+1)}..\tilde{\omega}_l^{(s)}
\end{aligned}
$$

With 7.4 and Lemma 7.9 we find that

$$
\lim_{\varepsilon\to 0}\left\{ \varepsilon\!\!\!\int\limits_\gamma (f\varphi_2)\varphi_3\cdots\varphi_s - \varepsilon\!\!\!\int\limits_\gamma \varphi_1\ldots\varphi_s \right\} = 0.
$$

ad (ii) In the case $1<\lambda<s$, i. e. $\varphi_\lambda = df$, we go right into the computation.

$$
\begin{aligned}
&\varepsilon\!\!\!\int\limits_\gamma \varphi_1\cdots\varphi_{\lambda-1}df\varphi_{\lambda+1}\cdots\varphi_s \\
&= \sum_{0\leq\alpha\leq\beta\leq s} \int\limits_{\gamma\leq\tau_0-\epsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)} \int\limits_{[0;1]} \eta_\varepsilon^{(\alpha+1)}..\eta_\varepsilon^{(\beta)} \int\limits_{\gamma\geq\tau_0+\epsilon} \tilde{\omega}_l^{(\beta+1)}..\tilde{\omega}_l^{(s)} \\
&= \sum_{\lambda+1\leq\alpha\leq\beta\leq s} \int\limits_{\gamma\leq\tau_0-\epsilon} \tilde{\omega}_k^{(1)}\cdots dG_k\cdots\tilde{\omega}_k^{(\alpha)} \int\limits_{[0;1]} \eta_\varepsilon^{(\alpha+1)}..\eta_\varepsilon^{(\beta)} \int\limits_{\gamma\geq\tau_0+\epsilon} \tilde{\omega}_l^{(\beta+1)}..\tilde{\omega}_l^{(s)} \\
&+ \sum_{\lambda=\alpha\leq\beta\leq s} \int\limits_{\gamma\leq\tau_0-\epsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\lambda-1)}dG_k \int\limits_{[0;1]} \eta_\varepsilon^{(\alpha+1)}..\eta_\varepsilon^{(\beta)} \int\limits_{\gamma\geq\tau_0+\epsilon} \tilde{\omega}_l^{(\beta+1)}..\tilde{\omega}_l^{(s)} \\
&+ \sum_{\substack{0\leq\alpha\leq\lambda-2\\ \lambda+1\leq\beta\leq s}} \int\limits_{\gamma\leq\tau_0-\epsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)} \int\limits_{[0;1]} \eta_\varepsilon^{(\alpha+1)}\cdots dP_\varepsilon\cdots\eta_\varepsilon^{(\beta)} \int\limits_{\gamma\geq\tau_0+\epsilon} \tilde{\omega}_l^{(\beta+1)}..\tilde{\omega}_l^{(s)} \\
&+ \sum_{\substack{\alpha=\lambda-1\\ \lambda\leq\beta\leq s}} \int\limits_{\gamma\leq\tau_0-\epsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)} \int\limits_{[0;1]} dP_\varepsilon\eta_\varepsilon^{(\lambda+1)}\cdots\eta_\varepsilon^{(\beta)} \int\limits_{\gamma\geq\tau_0+\epsilon} \tilde{\omega}_l^{(\beta+1)}..\tilde{\omega}_l^{(s)} \\
&+ \sum_{\substack{0\leq\alpha\leq\lambda-2\\ \lambda=\beta}} \int\limits_{\gamma\leq\tau_0-\epsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)} \int\limits_{[0;1]} \eta_\varepsilon^{(\alpha+1)}\cdots\eta_\varepsilon^{(\lambda-1)}dP_\varepsilon \int\limits_{\gamma\geq\tau_0+\epsilon} \tilde{\omega}_l^{(\beta+1)}..\tilde{\omega}_l^{(s)} \\
&+ \sum_{0\leq\alpha\leq\beta+1=\lambda} \int\limits_{\gamma\leq\tau_0-\epsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)} \int\limits_{[0;1]} \eta_\varepsilon^{(\alpha+1)}\cdots\eta_\varepsilon^{(\lambda-1)} \int\limits_{\gamma\geq\tau_0+\epsilon} dG_l\tilde{\omega}_l^{(\lambda+1)}\cdots\tilde{\omega}_k^{(s)} \\
&+ \sum_{0\leq\alpha\leq\beta\leq\lambda-2} \int\limits_{\gamma\leq\tau_0-\epsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)} \int\limits_{[0;1]} \eta_\varepsilon^{(\alpha+1)}..\eta_\varepsilon^{(\beta)} \int\limits_{\gamma\geq\tau_0+\epsilon} \tilde{\omega}_l^{(\beta+1)}\cdots dG_l\tilde{\omega}_l^{(s)}
\end{aligned}
$$

$$= \sum_{\lambda+1\le\alpha\le\beta\le s} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots(G_k\tilde{\alpha}_k^{(\lambda+1)})\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdot\cdot\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$+ \sum_{1+\lambda=\alpha\le\beta\le s} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\lambda-1)}\cdot G_k(\gamma_x(\tau_0-\varepsilon))\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdot\cdot\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$+ \sum_{\substack{0\le\alpha\le\lambda-2\\ \lambda+1\le\beta\le s}} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdots(P_\varepsilon\eta_\varepsilon^{(\lambda+1)})\cdots\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$+ \sum_{\substack{\alpha=\lambda-1\\ \lambda\le\beta\le s}} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}(P_\varepsilon\eta_\varepsilon^{(\lambda+1)})\cdots\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$+ \sum_{\substack{0\le\alpha\le\lambda-2\\ \lambda=\beta}} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdots\eta_\varepsilon^{(\lambda-1)}\cdot P_\varepsilon(1)\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$+ \sum_{0\le\alpha\le\beta+1=\lambda} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdots\eta_\varepsilon^{(\lambda-1)}\int_{\gamma\ge\tau_0+\varepsilon}(G_l\tilde{\omega}_l^{(\lambda+1)})\cdots\tilde{\omega}_k^{(s)}$$

$$+ \sum_{0\le\alpha\le\beta\le\lambda-2} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdot\cdot\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdots(G_l\tilde{\omega}_l^{(\lambda+1)})\cdots\tilde{\omega}_l^{(s)}$$

$$- \sum_{\lambda+1\le\alpha\le\beta\le s} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots(G_k\tilde{\omega}_k^{(\lambda-1)})\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdot\cdot\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$- \sum_{\lambda=\alpha\le\beta\le s} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots(G_k\tilde{\omega}_k^{(\lambda-1)})\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdot\cdot\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$- \sum_{\substack{0\le\alpha\le\lambda-2\\ \lambda+1\le\beta\le s}} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdots(P_\varepsilon\eta_\varepsilon^{(\lambda-1)})\cdots\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$- \sum_{\substack{\alpha=\lambda-1\\ 1+\lambda\le\beta\le s}} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}P_\varepsilon(0)\cdot\int_{[0;1]}\eta_\varepsilon^{(\lambda+1)}\cdots\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$- \sum_{\substack{0\le\alpha\le\lambda-2\\ \lambda=\beta}} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdots(P_\varepsilon\eta_\varepsilon^{(\lambda-1)})\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdot\cdot\tilde{\omega}_l^{(s)}$$

$$- \sum_{0\le\alpha\le\lambda-2} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdots\eta_\varepsilon^{(\lambda-1)}G_l(\gamma_y(\tau_0+\varepsilon))\cdot\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\lambda+1)}\cdots\tilde{\omega}_k^{(s)}$$

$$- \sum_{0\le\alpha\le\beta\le\lambda-2} \int_{\gamma\le\tau_0-\varepsilon} \tilde{\omega}_k^{(1)}\cdots\tilde{\omega}_k^{(\alpha)}\int_{[0;1]}\eta_\varepsilon^{(\alpha+1)}\cdot\cdot\eta_\varepsilon^{(\beta)}\int_{\gamma\ge\tau_0+\varepsilon}\tilde{\omega}_l^{(\beta+1)}\cdots(G_l\tilde{\omega}_l^{(\lambda-1)})\cdots\tilde{\omega}_l^{(s)}$$

Now we collect all those sums in which one of the four constants $\{G_k(\gamma_x(\tau_0-\varepsilon)),\ P_\varepsilon(1),\ P_\varepsilon(0)\ G_l(\gamma_y(\tau_0+\varepsilon))\}$ appears.

$$\begin{aligned}
&= (G_k(\gamma_x(\tau_0-\varepsilon)) - P_\varepsilon(0)) \\
&\cdot \left(\sum_{\lambda+1=\alpha+1\le\beta\le s} \int_{\gamma\le\tau_0-\epsilon} \tilde\omega_k^{(1)}\cdots\tilde\omega_k^{(\lambda-1)} \int_{[0;1]} \eta_\varepsilon^{(\alpha+1)}\cdots\eta_\varepsilon^{(\beta)} \int_{\gamma\ge\tau_0+\epsilon} \tilde\omega_l^{(\beta+1)}\cdots\tilde\omega_l^{(s)} \right) \\
&+ (P_\varepsilon(1) - G_l(\gamma_y(\tau_0+\varepsilon))) \\
&\cdot \left(\sum_{0\le\alpha\le\lambda-2} \int_{\gamma\le\tau_0-\epsilon} \tilde\omega_k^{(1)}\cdots\tilde\omega_k^{(\alpha)} \int_{[0;1]} \eta_\varepsilon^{(\alpha+1)}\cdots\eta_\varepsilon^{(\lambda-1)} \int_{\gamma\ge\tau_0+\epsilon} \tilde\omega_l^{(\lambda+1)}\cdots\tilde\omega_k^{(s)} \right) \\
&+ \mathcal{E}\!\!\int_\gamma \varphi_1\cdots(f\varphi_{\lambda+1})\cdots\varphi_s - \mathcal{E}\!\!\int_\gamma \varphi_1\cdots(f\varphi_{\lambda-1})\cdots\varphi_s
\end{aligned}$$

When we apply 7.4 and Lemma 7.9 we find:

$$\lim_{\varepsilon\to0}\left\{ \mathcal{E}\!\!\int_\gamma \varphi_1\cdots\varphi_s - \mathcal{E}\!\!\int_\gamma \varphi_1\cdots(f\varphi_{\lambda+1})\cdots\varphi_s + \mathcal{E}\!\!\int_\gamma \varphi_1\cdots(f\varphi_{\lambda-1})\cdots\varphi_s \right\} = 0.$$

ad (iii) Assume now $f(q) = f(\gamma(b)) = 0$ and use the second part of Proposition 1.3 to show:

$$\mathcal{E}\!\!\int_\gamma \varphi_1\cdots(f_s\varphi_{s-1}) = (-1)^{s-1}\mathcal{E}\!\!\int_{\gamma^{-1}} (f_s\varphi_{s-1})\cdots\varphi_1$$

and

$$\mathcal{E}\!\!\int_\gamma \varphi_1\cdots\varphi_s = (-1)^s\mathcal{E}\!\!\int_{\gamma^{-1}} \varphi_s\cdots\varphi_1.$$

Then apply (i).

A path colliding once with the double point.

Now let $\gamma : [a;b] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ be a path over $\vec{v}$ that meets the double point p_{kl} once with parameter value $\tau_0 \in]a;b[$ and stays in one component D_k.

ad(i) If $\lambda = 1$ assume again $f(p) = f(\gamma(a)) = 0$. Then we compute as follows.

$$\begin{aligned}
&\mathcal{E}\!\!\int_\gamma (f\varphi_2)\varphi_3\cdots\varphi_s \\
&= \int_{\gamma\ge\tau_0+\epsilon} (G_k\tilde\omega_k^{(2)})\tilde\omega_k^{(3)}\cdots\tilde\omega_k^{(s)} \\
&+ \sum_{1\le\alpha\le s} \int_{\gamma\le\tau_0-\epsilon} (G_k\tilde\omega_k^{(2)})\tilde\omega_k^{(3)}\cdots\tilde\omega_k^{(\alpha)} \int_{\gamma\ge\tau_0+\epsilon} \tilde\omega_k^{(\alpha+1)}\cdots\tilde\omega_k^{(s)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{0\le\alpha\le s}\ \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)}\cdots\tilde\omega_k^{(\alpha)} \int\limits_{\gamma\ge\tau_0+\varepsilon} \tilde\omega_k^{(\alpha+1)}\cdots\tilde\omega_k^{(s)} \\
&+ \left(G_k(\gamma_x(\tau_0+\varepsilon)) - \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)} \right) \\
&= \varepsilon\!\!\!\int_\gamma (f\varphi_2)\varphi_3\cdots\varphi_s + (G_k(\gamma_x(\tau_0+\varepsilon)) - G_k(\gamma(\tau_0-\varepsilon)))\,.
\end{aligned}
$$

This shows by 7.4 and Lemma 7.9:

$$
\lim_{\varepsilon\to 0}\left\{ \varepsilon\!\!\!\int_\gamma (f\varphi_2)\varphi_3\cdots\varphi_s - \varepsilon\!\!\!\int_\gamma \varphi_1\ldots\varphi_s \right\} = 0.
$$

ad (ii) In the case $1<\lambda<s$ we proceed as follows.

$$
\begin{aligned}
&\varepsilon\!\!\!\int_\gamma \varphi_1\cdots\varphi_{\lambda-1}df\varphi_{\lambda+1}\cdots\varphi_s \\
&= \sum_{\lambda+1\le\alpha\le s}\ \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)}\cdots dG_k\cdots\tilde\omega_k^{(\alpha)} \int\limits_{\gamma\ge\tau_0+\varepsilon} \tilde\omega_k^{(\alpha+1)}\cdots\tilde\omega_k^{(s)} \\
&+ \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)}\cdots\tilde\omega_k^{(\lambda-1)}dG_k \int\limits_{\gamma\ge\tau_0+\varepsilon} \tilde\omega_k^{(\alpha+1)}\cdots\tilde\omega_k^{(s)} \\
&+ \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)}\cdots\tilde\omega_k^{(\alpha)} \int\limits_{\gamma\ge\tau_0+\varepsilon} dG_k\tilde\omega_k^{(\lambda+1)}\cdots\tilde\omega_k^{(s)} \\
&+ \sum_{0\le\alpha\le\lambda-2}\ \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)}\cdots\tilde\omega_k^{(\alpha)} \int\limits_{\gamma\ge\tau_0+\varepsilon} \tilde\omega_k^{(\alpha+1)}\cdots dG_k\tilde\omega_k^{(s)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\lambda+1\le\alpha\le s}\ \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)}\cdots(G_k\tilde\alpha_k^{(\lambda+1)})\cdots\tilde\omega_k^{(\alpha)} \int\limits_{\gamma\ge\tau_0+\varepsilon} \tilde\omega_k^{(\alpha+1)}\cdots\tilde\omega_k^{(s)} \\
&+ \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)}\cdots\tilde\omega_k^{(\lambda-1)}\cdot G_k(\gamma_x(\tau_0-\varepsilon)) \int\limits_{\gamma\ge\tau_0+\varepsilon} \tilde\omega_k^{(\lambda+1)}\cdots\tilde\omega_k^{(s)} \\
&+ \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)}\cdots\tilde\omega_k^{(\lambda-1)} \int\limits_{\gamma\ge\tau_0+\varepsilon} (G_k\tilde\omega_k^{(\lambda+1)})\cdots\tilde\omega_k^{(s)} \\
&+ \sum_{0\le\alpha\le\lambda-2}\ \int\limits_{\gamma\le\tau_0-\varepsilon} \tilde\omega_k^{(1)}\cdots\tilde\omega_k^{(\alpha)} \int\limits_{\gamma\ge\tau_0+\varepsilon} \tilde\omega_k^{(\alpha+1)}\cdots(G_k\tilde\omega_k^{(\lambda+1)})\cdots\tilde\omega_k^{(s)}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\lambda+1 \le \alpha \le s} \int_{\gamma \le \tau_0 - \varepsilon} \tilde{\omega}_k^{(1)} \cdots (G_k \tilde{\alpha}_k^{(\lambda-1)}) \cdots \tilde{\omega}_k^{(\alpha)} \int_{\gamma \ge \tau_0 + \varepsilon} \tilde{\omega}_k^{(\alpha+1)} \cdots \tilde{\omega}_k^{(s)} \\
&- \int_{\gamma \le \tau_0 - \varepsilon} \tilde{\omega}_k^{(1)} \cdots G_k \tilde{\omega}_k^{(\lambda-1)} \int_{\gamma \ge \tau_0 + \varepsilon} \tilde{\omega}_k^{(\lambda+1)} \cdots \tilde{\omega}_k^{(s)} \\
&- \int_{\gamma \le \tau_0 - \varepsilon} \tilde{\omega}_k^{(1)} \cdots \tilde{\omega}_k^{(\lambda-1)} \cdot G_k(\gamma_x(\tau_0 + \varepsilon)) \int_{\gamma \ge \tau_0 + \varepsilon} (G_k \tilde{\omega}_k^{(\lambda+1)}) \cdots \tilde{\omega}_k^{(s)} \\
&- \sum_{0 \le \alpha \le \lambda - 2} \int_{\gamma \le \tau_0 - \varepsilon} \tilde{\omega}_k^{(1)} \cdots \tilde{\omega}_k^{(\alpha)} \int_{\gamma \ge \tau_0 + \varepsilon} \tilde{\omega}_k^{(\alpha+1)} \cdots (G_k \tilde{\omega}_k^{(\lambda-1)}) \cdots \tilde{\omega}_k^{(s)}
\end{aligned}
$$

$$
\begin{aligned}
&= (G_k(\gamma_x(\tau_0 - \varepsilon)) - G_k(\gamma_x(\tau_0 + \varepsilon))) \\
&\quad \cdot \left(\int_{\gamma \le \tau_0 - \varepsilon} \tilde{\omega}_k^{(1)} \cdots \tilde{\omega}_k^{(\lambda-1)} \int_{\gamma \ge \tau_0 + \varepsilon} \tilde{\omega}_k^{(\lambda+1)} \cdots \tilde{\omega}_k^{(s)} \right) \\
&\quad + \varepsilon \int_\gamma \varphi_1 \cdots (f \varphi_{\lambda+1}) \cdots \varphi_s - \varepsilon \int_\gamma \varphi_1 \cdots (f \varphi_{\lambda-1}) \cdots \varphi_s
\end{aligned}
$$

By 7.4 and Lemma 7.9 we conclude:

$$
\lim_{\varepsilon \to 0} \left\{ \varepsilon \int_\gamma \varphi_1 \cdots \varphi_s - \varepsilon \int_\gamma \varphi_1 \cdots (f \varphi_{\lambda+1}) \cdots \varphi_s + \varepsilon \int_\gamma \varphi_1 \cdots (f \varphi_{\lambda-1}) \cdots \varphi_s \right\} = 0.
$$

ad (iii) Assume now $f(q) = f(\gamma(b)) = 0$ and use again the second part of Proposition 1.3 to show:

$$
\varepsilon \int_\gamma \varphi_1 \cdots (f_s \varphi_{s-1}) = (-1)^{s-1} \varepsilon \int_{\gamma^{-1}} (f_s \varphi_{s-1}) \cdots \varphi_1
$$

and
$$
\varepsilon \int_\gamma \varphi_1 \cdots \varphi_s = (-1)^s \varepsilon \int_{\gamma^{-1}} \varphi_s \cdots \varphi_1 .
$$

Then apply (i).

A path locally at a double point Assume now that $\gamma : [a; b] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ is a path over $\vec{v}$. Note that we can decompose γ into pieces each of which meets the double point once:

$$
\gamma = \gamma_1 \star \cdots \star \gamma_N .
$$

We saw already that each of the paths γ_i with $i = 1, \dots, N$ has the the following property with respect to $\varphi_1 \otimes \cdots \otimes \varphi_s$ and λ, which we refer to for the moment as *property P*.

We say that a path $\gamma : [a; b] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ *over* $\vec{v}$ *has property P with respect to* $\varphi_1 \otimes \cdots \otimes \varphi_s$ *(where* $\varphi_\lambda = df_\lambda$*) iff it satisfies – depending on* λ *– one of the three equations:*

$$\lim_{\varepsilon \to 0} \left\{ \oint_\gamma df_1 \varphi_2 \ldots \varphi_s - \oint_\gamma ((f_1 - f_1(\gamma(a))\varphi_2)\varphi_3 \cdots \varphi_s \right\} = 0$$

$$\lim_{\varepsilon \to 0} \left\{ \oint_\gamma \varphi_1 \cdots df_\lambda \cdots \varphi_s - \oint_\gamma \varphi_1 \cdots (f_\lambda \varphi_{\lambda+1}) \cdots \varphi_s + \oint_\gamma \varphi_1 \cdots (f_\lambda \varphi_{\lambda-1}) \cdots \varphi_s \right\} = 0$$

$$\lim_{\varepsilon \to 0} \left\{ \oint_\gamma \varphi_1 \cdots \varphi_{s-1} df_s - \oint_\gamma \varphi_1 \cdots \varphi_{s-2}(f_s - f_s(\gamma(b))\varphi_{s-1}) \right\} = 0$$

and $\int_\gamma \varphi_\lambda = f_\lambda(\gamma(b)) - f_\lambda(\gamma(a))$.

It remains to show that if a path $\alpha : [0; 1] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ over $\vec{v}$ and a path $\beta : [0; 1] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ over $\vec{v}$ with $\alpha(1) = \beta(0)$ both have property P then also their composition $\alpha \star \beta$ has property P.

This is a straightforward calculation similar to those on pages 122 – 128, which finally makes again use of 7.4 and 7.9. □

7.2.3 The Global Case

In this subsection we consider the global situation of paths $\gamma : [0; 1] \to Z_0$ over $\vec{v}$ again. Our goal is to establish the fact that the iterated integral along a path γ over $\vec{v}$, which is composed of several such paths, can be computed from iterated integrals over these parts of the path γ.

Remark 7.10 We prove these things directly. There might be more elegant ways to derive properties of *global iterated integrals* from local ones by using Chen's operator T (cf. [Che77a], [Hai87c]). However, for the moment we confine ourselves to the direct approach.

First, let us formalize for the global case, what we already observed locally in the proof of Theorem 7.7.

Lemma 7.11 *Let* $\varphi_1, \ldots, \varphi_s \in A^1$ *such that one of these forms, say* φ_λ*, is exact:* $\varphi_\lambda = df_\lambda$*. Moreover let* $\alpha : [0; 1] \to Z_0$ *and* $\beta : [a; b] \to Z_0$ *be paths over* $\vec{v}$ *with* $\alpha(1) = \beta(0)$ *such that for* $\gamma = \alpha$ *and* $\gamma = \beta$ *holds:*

If $\lambda = 1$ *then* $\lim_{\varepsilon \to 0} \left\{ \oint_\gamma df_1 \varphi_2 \ldots \varphi_s - \oint_\gamma ((f_1 - f_1(\gamma(a))\varphi_2)\varphi_3 \cdots \varphi_s \right\} = 0,$
or if $2 \leq \lambda \leq s - 1$ *then*

$$\lim_{\varepsilon \to 0} \left\{ \oint_\gamma \varphi_1 \cdots df_\lambda \cdots \varphi_s - \oint_\gamma \varphi_1 \cdots (f_\lambda \varphi_{\lambda+1}) \cdots \varphi_s + \oint_\gamma \varphi_1 \cdots (f_\lambda \varphi_{\lambda-1}) \cdots \varphi_s \right\} = 0$$

or if $\lambda = s$ then

$$\lim_{\varepsilon \to 0} \left\{ \varepsilon\!\!\int_\gamma \varphi_1 \cdots \varphi_{s-1} df_s - \varepsilon\!\!\int_\gamma \varphi_1 \cdots \varphi_{s-2}\big(f_s - f_s(\gamma(b))\varphi_{s-1}\big) \right\} = 0,$$

<u>and</u> it holds (whatever λ is): $\int\limits_\gamma \varphi_\lambda = f_\lambda(\gamma(b)) - f_\lambda(\gamma(a))$.

Then the above equations are also satisfied for $\gamma = \alpha \star \beta$. □

Again, the proof of this Lemma is a direct computation similar to those on the pages 122 - 128.

Our next goal is to establish the fact that the knowledge of iterated integrals along all parts of a path over $\vec{v}$ implies the knowledge of iterated integrals along the entire path.

Lemma 7.12 *Let p, q be two points in $Z_0 \setminus \bigcup_{[k<l]}\{p_{kl}\}$. There exists for any $I = \sum_{|J| \leq s} a_J\ \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1$ a finite number $I_1^{p,q}, \ldots, I_M^{p,q}$ of Chen-closed elements in $\bigoplus_{r=1}^{s} \bigotimes^r A^1$ and a function $F : \mathbb{C}^{2M} \to \mathbb{C}$ such that for two paths α, $\beta : [0;1] \to Z_0$ over $\vec{v}$ with: $\alpha(0) = p$, $\alpha(1) = \beta(0)$ and $\beta(1) = \beta'(0) = q$ holds:*

$$\int\limits_{\alpha\star\beta} I = F\left(\int\limits_\alpha I_1^{p,q}, \ldots, \int\limits_\alpha I_M^{p,q}, \int\limits_\beta I_1^{p,q}, \ldots, \int\limits_\beta I_M^{p,q} \right).$$

Corollary 7.13 *Let p, q, r be three points in $Z_0 \setminus \bigcup_{[k<l]}\{p_{kl}\}$. Suppose α, α', β, $\beta' : [0;1] \to Z_0$ are four paths over $\vec{v}$ with: $\alpha(0) = \alpha'(0) = p$, $\alpha(1) = \alpha'(1) = \beta(0) = \beta'(0) = q$ and $\beta(1) = \beta'(0) = r$.*

If we have for all $I = \sum_{|J| \leq s} a_J\ \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1$

$$\int\limits_\alpha I = \int\limits_{\alpha'} I \quad \textit{and} \quad \int\limits_\beta I = \int\limits_{\beta'} I,$$

then also for all $I = \sum_{|J| \leq s} a_J\ \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1$ holds:

$$\int\limits_{\alpha\star\beta} I = \int\limits_{\alpha'\star\beta'} I.$$

Proof of 7.12: We first prove the lemma for a special type of I.

Let $\varphi_1, \ldots, \varphi_s$ be closed forms in A^1. According to Proposition 1.11 there exist forms $\varphi_{i,i+1,\ \ ,j} \in A^1$ with $1 \leq i < j \leq s$ such that $(\varphi_{i,\ \ ,i} = \varphi_i)$

$$d\varphi_{i,i+1,\ \ ,j} + \sum_{k=i}^{j-1} \varphi_{i,\ \ ,k} \wedge \varphi_{k+1,\ \ ,j} = 0$$

and

$$S_{1,\quad,s} = \sum_{r=1}^{s} \sum_{0<\alpha_1<\quad<\alpha_{r-1}<s} \varphi_{1,\quad,\alpha_1} \otimes \varphi_{\alpha_1+1,\quad,\alpha_2} \otimes \cdots \otimes \varphi_{\alpha_{r-1}+1,\quad,s}$$

is Chen-closed. Let us refer to such an element as to a *standard Chen-closed element for* $\varphi_1, \ldots, \varphi_s$.

The first step is to prove the lemma for standard Chen-closed elements. Consider $\oint_{\alpha\star\beta} S_{1,\quad,s}$. This is by Proposition 7.1 equal to

$$\begin{aligned}
&\oint_{\alpha\star\beta} S_{1,\quad,s} \\
&= \sum_{r=1}^{s} \sum_{0<\alpha_1<\quad<\alpha_{r-1}<s} \oint_{\alpha\star\beta} \varphi_{1,\quad,\alpha_1} \varphi_{\alpha+1,\quad\alpha_2} \cdots \varphi_{\alpha_{r-1}+1,\quad,s} \\
&= \sum_{r=1}^{s} \sum_{0\le m\le s} \sum_{0=\alpha_0<\alpha_1<\quad<\alpha_{r-1}<\alpha_r=s} \oint_{\alpha} \varphi_{1,\quad,\alpha_1} \cdots \varphi_{\alpha_{m-1}+1,\quad,\alpha_m} \\
&\qquad\qquad \cdot \oint_{\beta} \varphi_{\alpha_m+1,\quad,\alpha_{m+1}} \quad \cdot \varphi_{\alpha_{r-1}+1,\quad,\alpha_s} \\
&= \oint_{\alpha} S_{1,\quad,s} + \oint_{\beta} S_{1,\quad,s} \\
&+ \sum_{r=1}^{s} \sum_{0<m<s} \sum_{m\le\alpha_m\le s-r+m+1} \oint_{\alpha} S_{1,\quad,\alpha_m} \cdot \oint_{\beta} S_{\alpha_m+1,\quad,s},
\end{aligned}$$

where

$$S_{k,\quad,k+l} = \sum_{r=1}^{l} \sum_{k<\alpha_1<\quad<\alpha_{r-1}<k+l} \varphi_{k+1,\quad\alpha_1} \otimes \cdots \otimes \varphi_{\alpha_{r-1}+1,\quad,k+l}$$

for $1 \le k \le k+l \le s$ is a standard closed element for $\varphi_k \otimes \cdots \otimes \varphi_{k+l}$. This proves the lemma for standard Chen-closed elements.

Now we prove the lemma for general Chen-closed

$$I = \sum_{|J|\le s} a_J\, \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^{r} A^1$$

by induction on the maximal length s. First, replace for all J with $|J| = s$, where one of the φ_{j_λ} is exact, i. e. $\varphi_{j_\lambda} = df_{j_\lambda}$ for some $f_{j_\lambda} \in A^0$, the $\varphi_{j_1} \otimes \cdot \quad \cdot \otimes \varphi_{j_s}$ by

$$\Big(\big(f_{j_1} - f_{j_1}(p)\big)\varphi_{j_2}\Big)\varphi_{j_3} \cdots \varphi_{j_s},$$

or

$$\varphi_{j_1} \cdots (f_{j_\lambda} \varphi_{j_{\lambda+1}}) \cdots \varphi_{j_s} - \varphi_{j_1} \cdots (f_{j_\lambda} \varphi_{j_{\lambda-1}}) \cdots \varphi_{j_s},$$

or

$$\varphi_{j_1} \cdots \varphi_{j_{s-2}} \Big((f_{j_\lambda} - f_{j_\lambda}(q)) \varphi_{j_{s-1}} \Big),$$

depending on, whether $\lambda = 1$ or $2 \leq \lambda \leq s-1$ or $\lambda = s$. Call the result of this operation $\tilde{I} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1$ and note that it is Chen-closed.

Let $\gamma : [a, b] \to Z_0$ be some path over $\vec{v}$ from p to q, for instance $\gamma = \alpha \star \beta$. Note that we may decompose γ into pieces $\gamma = \gamma_1 \star \cdots \star \gamma_N$ each of which is a path over $\vec{v}$ and lies either in a $U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ or entirely in a component D_i. On each of these pieces the conditions of Lemma 7.11 are satisfied. Thus we find:

$$\int_\gamma I = \int_\gamma \tilde{I}$$

by virtue of this Lemma 7.11.

Knowing that the assertion is true, we may hence (by Lemma 1.10) assume that for each J, with $|J| = s$, all the φ_{j_m} are closed. Choose for each J with $|J| = s$ a standard Chen-closed element for $\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_s}$ and denote it by $S_{j_1, \dots, j_s}$.

Since $I - \sum_{|J|=s} a_J S_{j_1, \dots, j_s}$ is in $\bigoplus_{r=1}^{s-1} \bigotimes^r A^1$ we may apply the induction hypothesis. Together with the fact that we already proved the lemma for $\sum_{|J|=s} a_J S_{j_1, \dots, j_s}$, the proof can be accomplished easily. □

Theorem 7.14 *Let $\gamma : [0; 1] \to Z_0$ be a path over $\vec{v}$ and let*

$$I = \sum_{|J| \leq s} a_J \, \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^{r} A^1$$

be Chen-closed. Then $\int_\gamma I$ does not depend on the choice of coordinates (x, y) around the double points.

Proof: Due to Lemma 7.12 we may assume without loss of generality that the path $\gamma : [0; 1] \to Z_0$ over $\vec{v}$ lies entirely in a $U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ for $[k < l]$ and meets the double point p_{kl} once with parameter value τ_0. Like earlier we distinguish to cases.

A path traversing the double point

We have

$$\int_\gamma I = \lim_{\varepsilon \to 0} \left\{ \sum_J \overset{\varepsilon}{\int_\gamma} \varphi_{j_1} \cdots \varphi_{j_r} \right\}$$

Recall that for each J there are polynomials $Q^J_{\alpha\beta} \in \mathbb{C}[u]$ such that

$$\varepsilon\!\!\int_\gamma \varphi_{j_1} \cdots \varphi_{j_r}$$

$$= \sum_{0 \le \alpha \le \beta \le r} \int_{\gamma \le \tau_0 - \varepsilon} \tilde{\omega}_k^{(1)} \cdots \tilde{\omega}_k^{(\alpha)} Q^J_{\alpha\beta} \left(\log \gamma_x(\tau_0 - \varepsilon) \cdot \gamma_y(\tau_0 + \varepsilon)\right) \int_{\gamma \ge \tau_0 + \varepsilon} \tilde{\omega}_l^{(\beta+1)} \cdots \tilde{\omega}_l^{(s)}.$$

Furthermore remark:

$$(\log (\gamma_x(\tau_0 - \varepsilon) \cdot \gamma_y(\tau_0 + \varepsilon)))^m = \left(\log \left(\frac{\gamma_x(\tau_0 - \varepsilon) \cdot \gamma_y(\tau_0 + \varepsilon)}{\varepsilon^2}\right) + \log \varepsilon^2\right)^m$$

$$\begin{aligned}
&= \left(\log \varepsilon^2\right)^m \\
&+ \left\{ \log \left(\frac{\gamma_x(\tau_0 - \varepsilon) \cdot \gamma_y(\tau_0 + \varepsilon)}{\varepsilon^2}\right) \right. \\
&\left. \cdot \left(\sum_{k=1}^{m} \binom{m}{k} \log^{k-1} \left(\frac{\gamma_x(\tau_0 - \varepsilon) \cdot \gamma_y(\tau_0 + \varepsilon)}{\varepsilon^2}\right) \left(\log \varepsilon^2\right)^{m-k}\right) \right\}.
\end{aligned}$$

The function $\log\left(\frac{\gamma_x(\tau_0-\varepsilon)\cdot\gamma_y(\tau_0+\varepsilon)}{\varepsilon^2}\right)$ is a differentiable function in ε on an intervall $(-\varepsilon_0; \varepsilon_0)$ for some ε_0, which vanishes in 0. Hence the vanishing of this function is stronger than any logarithmic growth (Use Fact 7.4 on page 113). This indicates that the following limit is zero:

$$\lim_{\varepsilon \to 0} \left\{ \varepsilon\!\!\int_\gamma \varphi_{j_1} \cdots \varphi_{j_r} - \sum_{0 \le \alpha \le \beta \le r} \int_{\gamma \le \tau_0 - \varepsilon} \tilde{\omega}_k^{(1)} \cdots \tilde{\omega}_k^{(\alpha)} Q^J_{\alpha\beta} \left(\log \varepsilon^2\right) \int_{\gamma \ge \tau_0 + \varepsilon} \tilde{\omega}_l^{(\beta+1)} \cdots \tilde{\omega}_l^{(s)} \right\}.$$

We find that $\int_\gamma I$ does not depend on coordinates.

A path colliding with a double point

Let $p := \gamma(0)$ and $q := \gamma(1)$. We know by 7.7 that there exists a closed and hence exact $\varphi_{pq} = df_{pq} \in A^1_{kl}$ such that

$$\int_\gamma I = \int_\gamma \varphi_{pq} = f_{pq}(q) - f_{pq}(p).$$

Let $\check{\gamma} : [a; b] \to U^{k-}_{kl}$ be a path (over $\vec{v}$), which is nearby homotopic relative to the end points with γ, but does not touch the double point. According to Proposition 7.6 we observe:

$$\int_\gamma \varphi_{pq} = f_{pq}(q) - f_{pq}(p) = \int_{\check{\gamma}} \varphi_{pq}$$

The integral on the right is independent of the choice of coordinates. □

7.3 Nearby Homotopy Functionals

In this section we show the analogue of Theorem 1.13 on page 22. More precisely, we show that iterated integrals of Chen-closed elements in $\bigoplus_{r=1}^{s} \bigotimes^r A^1$ are invariant under a homotopy over $\vec{v}$ based at $\vec{w}$. Hence, they are *nearby homotopy functionals*. There is no converse of this assertion since the iterated integrals only exist for Chen-closed elements in $\bigoplus_{r=1}^{s} \bigotimes^r A^1$.

Theorem 7.15 *Let $I = \sum_{|J| \le s} a_J \; \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1$ be Chen-closed. Then for two paths $\gamma_0, \gamma_1 : [0;1] \to Z_0$ over $\vec{v}$ based at $\vec{w}$, which are nearby homotopic*

$$\int_{\gamma_0} I = \int_{\gamma_1} I.$$

Proof: Let $H : [0;1] \times [0;1] \to Z_0$ be the nearby homotopy. That is, $H(\cdot, 0) = \gamma_0$ and $H(\cdot, \gamma_1) = \gamma_1$ and for any $\lambda \in [0;1]$ is $H(\cdot, \lambda)$ is a path over $\vec{v}$ based at $\vec{w}$. Consider the function:

$$\begin{array}{rccc} c: & [0;1] & \longrightarrow & \mathbb{C} \\ & \lambda & \longmapsto & \int\limits_{H(\ ,\lambda)} I. \end{array}$$

We will show that c is constant by showing that it is locally constant.

Let $\lambda_0 \in [0;1]$. We show that c is locally constant at λ_0. Write $\eta := H(\cdot, \lambda_0)$ and denote by $0 = \tau_0 < \tau_1 < \cdots < \tau_N < \tau_{n+1} = 1$ the parameter values, where η meets the set of double points: η defines a map

$$\mu : \{0, \ldots, N+1\} \longrightarrow \{(k,l) | \; [k < l]\},$$

by $\eta(\tau_i) = p_{\mu(i)}$ for $i = 0, \ldots, N+1$.

For any $\varepsilon > 0$ we define $L_0^\varepsilon := [0; \varepsilon]$, $L_i^\varepsilon := [\tau_i - \varepsilon; \tau_i + \varepsilon]$ for $i = 1, \ldots, N$ and $L_{N+1}^\varepsilon := [1 - \varepsilon; 1]$. Choose a positive $\varepsilon_1 < \min\limits_i \frac{|\tau_{i+1} - \tau_i|}{2}$ (then all the $L_i^{\varepsilon_1}$ are mutually disjoint) such that:

$$\eta\left(L_i^{\varepsilon_1}\right) \subset U_{\mu(i)}^{k-} \cup \{p_{\mu(i)}\} \cup U_{\mu(i)}^{l-}.$$

By the continuity of H and the fact that all $H(\cdot, \lambda)$ are paths over $\vec{v}$ we can find a $\delta > 0$ and an $\varepsilon_0 > 0$ (with $\varepsilon_0 < \varepsilon_1$) such that for all $i = 0, \ldots, N+1$ holds:

$$H\left(L_i^{\varepsilon_0} \times [\lambda_0 - \delta; \lambda_0 + \delta]\right) \subset U_{\mu(i)}^{k-} \cup \{p_{\mu(i)}\} \cup U_{\mu(i)}^{l-}.$$

and the inverse image of the double points under $H|_{[0,1] \times [\lambda_0 - \delta, \lambda_0 + \delta]}$ lies entirely in the interior of the set $\bigcup_i L_i^{\varepsilon_0} \times [\lambda_0 - \delta, \lambda_0 + \delta]$. (If $\lambda_0 = 0$ resp. $\lambda_0 = 1$ then replace $[\lambda_0 - \delta, \lambda_0 + \delta]$ by $[0, \delta]$ resp. $[1 - \delta, 1]$.)

Fix a $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$. It is a consequence of Theorem 7.7 that $\int I$ takes the same value on any two paths $H \circ \theta_i$ $(i = 1, 2)$ over $\vec{v}$, where the θ_i are both

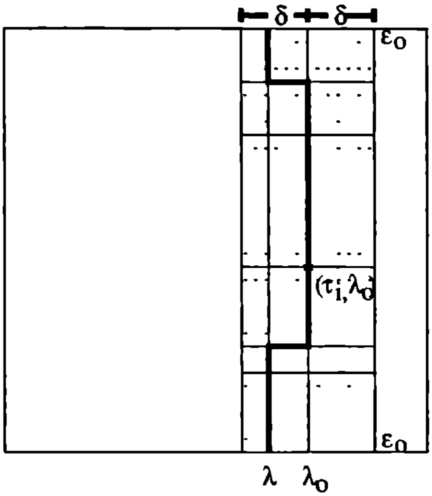


Figure 7.1: Illustration of the domain of H, the path η and the construction of the sequence of paths $(\alpha_i)_{i=0,\ldots,M}$.

paths with the same end points within $L_i^{\varepsilon_0} \times [\lambda_0 - \delta; \lambda_0 + \delta]$, whose end points are two of the four points: $\{(\tau_i \pm \varepsilon_0, \lambda_0),\ (\tau_i \pm \varepsilon_0, \lambda)\}$.

Moreover $\int I$ takes the same value on any two paths $H \circ \theta_i$ $(i = 1, 2)$ over $\vec{v}$, where the θ_i are both paths with the same end points within $[\tau_i + \varepsilon_0, \tau_{i+1} - \varepsilon_0] \times [\lambda_0 - \delta; \lambda_0 + \delta]$, whose end points are two of the four points: $\{(\tau_i + \varepsilon_0, \lambda_0),\ (\tau_{i+1} - \varepsilon_0, \lambda_0),\ (\tau_i + \varepsilon_0, \lambda),\ (\tau_{i+1} - \varepsilon_0, \lambda)\}$, likewise by Theorem 7.7.

Now one easily constructs a finite sequence of paths $\alpha_0, \alpha_1, \ldots, \alpha_M$ in $[0;1] \times [\lambda_0 - \delta; \lambda_0 + \delta]$ with the following properties:

- $H \circ \alpha_0 = H(\cdot, \lambda_0) = \eta$ and $H \circ \alpha_M = H(\cdot, \lambda)$,
- each $H \circ \alpha_i$ is a path over $\vec{v}$ based at $\vec{w}$,
- for two successive paths α_i, α_{i+1} holds that they coincide everywhere except either on an interval $L_i^{\varepsilon_0} \times [\lambda_0 - \delta; \lambda_0 + \delta]$ or on an interval $[\tau_i + \varepsilon_0, \tau_{i+1} - \varepsilon_0] \times [\lambda_0 - \delta; \lambda_0 + \delta]$.

In other words we may write

$$\alpha_i = \zeta \star \theta \star \vartheta \quad \text{and} \quad \alpha_{i+1} = \zeta \star \theta' \star \vartheta$$

and $\int I$ takes the same value on θ and θ'. Apply Corollary 7.13 to conclude $\int_{H \circ \alpha_i} I = \int_{H \circ \alpha_{i+1}} I$ □

7.3.1 Chen-Theorem for the Nearby Fundamental Group

First, let us prove the following analogue to Proposition 1.5.

Proposition 7.16 *Let $\alpha_1, \dots, \alpha_s$ be paths over $\vec{v}$ based at $\vec{w}$ and let $\varphi_1, \dots, \varphi_r \in A^1$ with $s \geq r$. Then*

$$\varepsilon\!\!\!\int_{(\alpha_1-1)\cdots(\alpha_s-1)} \varphi_1 \cdots \varphi_r = \begin{cases} \varepsilon\!\!\!\int_{\alpha_1} \varphi_1 \cdots \varepsilon\!\!\!\int_{\alpha_s} \varphi_s + V(\varepsilon), & \text{if } r = s \\ V(\varepsilon), & \text{if } r < s. \end{cases}$$

where V is a function $V : \mathbb{R}^{\geq 0} \to \mathbb{C}$ such that $\lim_{\varepsilon \to 0} V(\varepsilon) = 0$.

Proof: We write $A(\varepsilon) \equiv B(\varepsilon)$ if $A(\varepsilon) - B(\varepsilon) = V(\varepsilon)$ for a function $V(\varepsilon)$ with $\lim_{\varepsilon \to 0} V(\varepsilon) = 0$. One can prove (and this is easy) the following formula by induction on k:

$$\varepsilon\!\!\!\int_{(\alpha_1-1)\cdots(\alpha_k-1)\alpha_{k+1}\cdots\alpha_s} \varphi_1 \cdots \varphi_r$$

$$\equiv \sum_{\substack{k \leq N \leq s \\ 0<\nu_1< \ \ <\nu_k \leq \nu_{k+1} \leq \ \ \leq \nu_{N-1} \leq \nu_N = r}} \varepsilon\!\!\!\int_{\alpha_1} \varphi_1 \cdots \varphi_{\nu_1} \varepsilon\!\!\!\int_{\alpha_2} \varphi_{\nu_1+1} \cdots \varphi_{\nu_2} \cdots \varepsilon\!\!\!\int_{\alpha_N} \varphi_{\nu_{N-1}+1} \cdots \varphi_{\nu_N}.$$

For $k > r$, the sum is zero and for $k = r = s$ the sum is:

$$\varepsilon\!\!\!\int_{\alpha_1} \varphi_1 \cdots \varepsilon\!\!\!\int_{\alpha_s} \varphi_s.$$

□

Recall that we defined an augmentation $a : A^\bullet \to \mathbb{C}$ (see Definition 6.1 on page 91). Therefore Definition 1.7 of page 19 makes sense for the pair $(A^\bullet, a)$, whence we have the complex vector space

$$\boxed{H^0 \bar{B}_s(A^\bullet, a).}$$

Note in particular: $H^0 \bar{B}_1(A^\bullet, a) = H^1(A^\bullet)$.

By Theorem 7.15, Proposition 7.16 and by Lemma 7.11 there are well-defined integration maps

$$H^0 \bar{B}_s(A^\bullet, a) \to \mathrm{Hom}_{\mathbb{Z}}\left(\vec{J}/\vec{J}^{s+1}, \mathbb{C}\right).$$

The analogue for the nearby fundamental group of Chen's theorem or *the $\pi_1(Z_{\vec{v}}, \vec{w})$-De Rham-theorem* is the following.

Theorem 7.17 *Integration of Chen-closed elements in $\bigoplus_{r=1}^{s} \bigotimes^r A^1$ along paths over $\vec{v}$ based at $\vec{w}$ defines an isomorphism of complex vector spaces*

$$H^0 \bar{B}_s(A^\bullet, a) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}\left(J_{\vec{v}\vec{w}}/J_{\vec{v}\vec{w}}^{s+1}; \mathbb{C}\right).$$

Proof: We first indicate that for each $s \geq 1$ there is a short exact sequence

$$0 \to H^0\bar{B}_{s-1}(A^\bullet, a) \to H^0\bar{B}_s(A^\bullet, a) \to \bigotimes^s H^1(A^1) \to 0.$$

In particular, the map

$$\begin{array}{ccc} H^0\bar{B}_s(A^\bullet, a) & \to & \bigotimes^s H^1(A^1) \\ \sum_J a_J[\varphi_{j_1}|\cdots|\varphi_{j_r}] & \mapsto & \sum_J a_J[\varphi_{j_1}] \otimes \cdots \otimes [\varphi_{j_r}] \end{array}$$

is well-defined and is surjective by Proposition 1.11. Its kernel is $H^0\bar{B}_{s-1}(A^\bullet, a)$ by Lemma 1.10.

Then we have by Proposition 7.16 and Lemma 7.11 the commutative diagram:[3]

$$\begin{array}{ccccccccc} 0 & \to & H^0\bar{B}_{s-1}(A^\bullet, a) & \to & H^0\bar{B}_s(A^\bullet, a) & \to & \bigotimes^s H^1(A^1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (\vec{J}/\vec{J}^s)^* & \to & (\vec{J}/\vec{J}^{s+1})^* & \to & \bigotimes^s(\vec{J}/\vec{J}^2)^* & \to & 0 \end{array}$$

The theorem can now be proved by induction on s and the use of the 5-lemma. The beginning of the induction is given by Theorem 6.8. □

Warning 7.18 Assume that we are given two closed 1-forms φ, $\psi \in A^1$,

$$\varphi = \sum_{\imath \geq 0} 0 + \sum_{[k<l]} c_{kl} d\xi \text{ and } \psi = \sum_{\imath \geq 0} \omega_\imath + \sum_{[k<l]} c'_{kl} d\xi,$$

where all $\omega_\imath \in E^1(D_\imath)$. Then $\varphi \wedge \psi = 0$ and hence is $\int \varphi\psi$ a nearby homotopy functional. Define $\psi_0 = \sum_{i\geq 0} 0 + \sum_{[k<l]} c'_{kl} d\xi$. When we multiply φ and ψ (componentwise), we get the same as if we multiply φ and ψ_0. But nonetheless, in general holds: $\int \varphi\psi \neq \int \varphi\psi_0 = 0$. In general, there exists such φ and ψ for which there is a closed path α in one of the $D_\imath$'s such that $\int_\alpha \omega_\imath \neq 0$ and a path u over $\vec{v}$ starting at $\vec{w}$ and ending at $\alpha(0) = \alpha(1)$ such that $\int_u \varphi \neq 0$. Then

$$\begin{aligned} \int_{u * \alpha * u^{-1}} \varphi\psi &= \int_u \varphi \int_\alpha \psi - \int_\alpha \varphi \int_u \psi + \int_\alpha \varphi\psi \\ &= \int_u \varphi \int_\alpha \psi \end{aligned} \qquad \diamond$$

7.4 The Mixed Hodge Structure on $\vec{J}/\vec{J}^{s+1}$

After having developed this theory of iterated integrals along paths over $\vec{v}$, we are ready now to prove an analogue to Chen's Theorem 1.15 for the nearby fundamental group and finally to define a mixed Hodge Structure on $\vec{J}/\vec{J}^{s+1}$. With all what we know about iterated integrals along paths over $\vec{v}$ yet, this is just an application of standard techniques in mixed Hodge theory.

[3] The map $\bigotimes^s(\vec{J}/\vec{J}^2) \to \vec{J}^s/\vec{J}^{s+1}$, which is given by the group structure of $\pi_1(Z_{\vec{v}}, \vec{w})$, is an isomorphism since we have the isomorphism $\pi_1(Z_{\vec{v}}, \vec{w}) \cong \pi_1(Z_t, \sigma_p(t))$ for some $t \in \Delta^*$ and $p \in D_0^*$. In $\pi_1(Z_t, \sigma_p(t))$ it is true that $(J/J^{s+1})^* \to \bigotimes^s(J/J^2)^*$. is surjective.

7.4.1 The Hodge and the Weight Filtration

Let us use Theorem 7.17 to define a weight - and a Hodge filtration on $(\vec{J}/\vec{J}^{s+1})^*$.

First we define on $\bigoplus_{r=1}^{s} \bigotimes^{r} A^1$ a weight filtration $W_\bullet$ by

$$W_l\left(\bigoplus_{r=1}^{s}\bigotimes^{r} A^1\right) := \bigoplus_{\substack{r=1\\ l_1+\cdots+l_r+r\le l}}^{s} W_{l_1}A^1\otimes\cdots\otimes W_{l_r}A^1$$

and a Hodge filtration $F^\bullet$ by

$$F^p\left(\bigoplus_{r=1}^{s}\bigotimes^{r} A^1\right) := \bigoplus_{\substack{r=1\\ p_1+\cdots+p_r\ge p}}^{s} F^{p_1}A^1\otimes\cdots\otimes F^{p_r}A^1.$$

These filtrations $W_\bullet$ and $F^\bullet$ on $\bigoplus_{r=1}^{s}\bigotimes^{r} A^1$ induce filtrations $W_\bullet$ and $F^\bullet$ on $\bar{\mathcal{K}}^s(A^\bullet, a)$ and hence on $H^0\bar{B}_s(A^\bullet, a)$ in the following way:

$$\begin{aligned} W_l H^0\bar{B}_s(A^\bullet,a) &:= \operatorname{im}\left\{W_l\bar{\mathcal{K}}^s(A^\bullet,a)\to H^0\bar{B}_s(A^\bullet,a)\right\}\\ F^p H^0\bar{B}_s(A^\bullet,a) &:= \operatorname{im}\left\{F^p\bar{\mathcal{K}}^s(A^\bullet,a)\to H^0\bar{B}_s(A^\bullet,a)\right\}\end{aligned}$$

Proposition 7.19 *The short exact sequences (with $s \geq 2$)*

$$0\to H^0\bar{B}_{s-1}(A^\bullet,a)\xrightarrow{\iota} H^0\bar{B}_s(A^\bullet,a)\xrightarrow{pr}\bigotimes^{s} H^1(A^1)\to 0$$

are strict with respect to $W_\bullet$ and $F^\bullet$.

Proof: It is easy to see that pr is strict. We shall prove that ι is strict with respect to both filtrations.

Hodge filtration: Let $\bar{I} = \sum_J a_J\ [\varphi_{j_1}|\cdots|\varphi_{j_r}] \in F^pH^0\bar{B}_s(A^\bullet,a)\cap H^0\bar{B}_{s-1}(A^\bullet,a)$. Then there is a Chen-closed $I = \sum_{|J|\le s} a_J\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_r} \in F^p\bigoplus_{r=1}^{s}\bigotimes^{r} A^1$ such that $\int I = \int\bar{I}$. Since $\bar{I}\in H^0\bar{B}_{s-1}(A^\bullet,a)$ we may assume by Lemma 1.10, Lemma 7.11 and Proposition 7.16 that for J with $|J| = s$ all the $\varphi_{j_1},\dots,\varphi_{j_s}$ are exact: $\varphi_{j_m} = df_{j_m}$ and $f_{j_m}(p_{01}) = 0$. We have by definition:

$$\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_s}\in F^p\bigotimes^{s} A^1$$

$$:\Leftrightarrow \exists(p_1,\dots,p_s)\in\mathbb{Z}^s \text{ with } p_1+\cdots+p_s\ge p : \varphi_{j_1}\in F^{p_1},\dots,\varphi_{j_s}\in F^{p_s}.$$

Now there is either one $p_m \geq 1$ (which implies $\varphi_{j_m} = 0$ as $A^0 \xrightarrow{d} A^1$ is strict with respect to $F^\bullet$, i. e. $dA^0\cap F^1A^1 = 0$) or $p_1 = \cdots = p_s = 0$. Then

$$(f_{j_1} - a(f_{j_1}))\,\varphi_{j_2}\otimes\varphi_{j_3}\otimes\cdots\otimes\varphi_{j_s}\in F^p\bigotimes^{s-1} A^1.$$

Therefore, there is a Chen-closed $\tilde{I}\in F^p\bigoplus_{r=1}^{s-1}\bigotimes^{r} A^1$ such that $\int\tilde{I} = \int\bar{I}$.

Weight filtration: Let $\bar{I} \in W_l H^0 \bar{B}_s(A^\bullet, a) \cap H^0 \bar{B}_{s-1}(A^\bullet, a)$. Then there is a Chen-closed $I = \sum_{|J|\le s} a_J \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in W_l \bigoplus_{r=1}^{s} \bigotimes^r A^1$ such that $\int I = \int \bar{I}$. Because of $\bar{I} \in H^0 \bar{B}_{s-1}(A^\bullet, a)$ we may again assume by Lemma 1.10, Lemma 7.11 and Proposition 7.16 that for J with $|J| = s$ all the $\varphi_{j_1}, \dots, \varphi_{j_s}$ are exact: $\varphi_{j_m} = df_{j_m}$ and $f_{j_m}(p_{01}) = 0$.

Note that if $df \in W_\ell A^1$ then $f \in W_{\ell+1} A^0$. This can be seen as follows. Write $f = \sum_{i\ge 0} g_i + \sum_{[k<l]} P_0 + P_1 u + \cdots P_m u^m$. Now $f \in W_{\ell+1} \Leftrightarrow 2m \le \ell + 1$ and $df \in W_\ell \Leftrightarrow 2m - 1 \le \ell$.

We have by definition: $\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_s} \in W_l \bigotimes^s A^1$

$$:\Leftrightarrow \exists (l_1, \dots, l_s) \in \mathbb{Z}^s \text{ with } l_1 + \cdots + l_s + s \le l : \varphi_{j_1} \in W_{l_1}, \dots, \varphi_{j_s} \in W_{l_s}.$$

Then

$$(f_{j_1} - a(f_{j_1}))\, \varphi_{j_2} \otimes \varphi_{j_3} \otimes \cdots \otimes \varphi_{j_s} \in W_l \bigotimes^{s-1} A^1.$$

Hence, there is a Chen-closed $\tilde{I} \in W_l \bigoplus_{r=1}^{s-1} \bigotimes^r A^1$ such that $\int \tilde{I} = \int \bar{I}$. □

We finish this chapter by its main theorem.

Theorem 7.20 *For any $s \ge 1$ is*

$$\left(\vec{J}/\vec{J}^{s+1}\right)^* := \left(\left(\vec{J}/\vec{J}^{s+1}\right)^*_{\mathbb{Z}}, \left(\left(\vec{J}/\vec{J}^{s+1}\right)^*_{\mathbb{Q}}, W_\bullet\right) \left(\left(\vec{J}/\vec{J}^{s+1}\right)^*_{\mathbb{C}}, W_\bullet, F^\bullet\right)\right)$$

a mixed Hodge structure and the short exact sequence

$$0 \to \left(\vec{J}/\vec{J}^s\right)^* \to \left(\vec{J}/\vec{J}^{s+1}\right)^* \to H^1(A^\bullet)^{\otimes s} \to 0 \tag{7.6}$$

is a short exact sequence of MHSs.

Proof: We quote from [GS75], 1.16:

> Let $0 \to H' \to H \to H'' \to 0$ be an exact sequence of vector spaces. If two filtrations $W_\bullet$ and $F^\bullet$ for H induce mixed Hodge structures on both H' and H'', then they determine a mixed Hodge structure on H itself.

Since we know already Proposition 7.19, the proof of the theorem can be done by induction on s. □

7.5 The Monodromy on $\vec{J}/\vec{J}^{s+1}$

Recall that we defined in 6.4 the nilpotent chain-morphism

$$\begin{array}{rccc} N: & A^\bullet & \to & A^\bullet \\ & \phi & \mapsto & \frac{d}{d(-u)}\phi = -\frac{d}{du}\phi\,. \end{array}$$

with the properties $N(W_{l+1}A^\bullet) \subset W_{l-1}A^\bullet$ and $N(F^p A^\bullet) \subset F^{p-1}A^\bullet$.

Now we define a second nilpotent chain-morphism:

$$M : A^\bullet \to A^\bullet$$

by defining it first on $B^\bullet$ in the following way: Let M be the zero map on

$$\bigwedge\nolimits^\bullet(\frac{dp}{p}) \oplus \bigoplus_{i>0} E^\bullet(D_i \log P_i) \oplus \bigoplus_{[0<k<l]} E^\bullet(\Delta^1) \otimes \Lambda^\bullet.$$

Only on the one component $E^\bullet(\Delta^1) \otimes \Lambda^\bullet$ for $(k,l) = (0,1)$ it is, like N, defined by

$$\begin{array}{rccc} M: & E^\bullet(\Delta^1) \otimes \Lambda^\bullet & \to & E^\bullet(\Delta^1) \otimes \Lambda^\bullet \\ & \Xi & \mapsto & \frac{d}{d(-u)}\Xi = -\frac{d}{du}\Xi\,. \end{array}$$

Also M satisfies $M(W_{l+1}A^\bullet) \subset W_{l-1}A^\bullet$ and $M(F^pA^\bullet) \subset F^{p-1}A^\bullet$.

Remark 7.21 Note that for a $Q(u) \in E^\bullet(\Delta^1) \otimes \bigwedge^\bullet(\frac{dx}{x}, \frac{dy}{y})[u]$ and for a $\lambda \in \mathbb{C}^*$ always holds:

$$\lambda^N Q(u) = Q(u + \log\lambda).$$

Similar for M. Moreover, N and M define linear maps

$$N,\ M : \bigoplus_{r=1}^{s} \bigotimes^{r} A^\bullet \to \bigoplus_{r=1}^{s} \bigotimes^{r} A^\bullet, \tag{7.7}$$

which are defined on a generator $\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}$ by the Leibniz rule:

$$N\left(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}\right) = \sum_{\nu=1}^{r} \varphi_{j_1} \otimes \cdots \otimes N\varphi_{j_\nu} \otimes \cdots \otimes \varphi_{j_r} \quad \text{resp.}$$
$$M\left(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}\right) = \sum_{\nu=1}^{r} \varphi_{j_1} \otimes \cdots \otimes M\varphi_{j_\nu} \otimes \cdots \otimes \varphi_{j_r}.$$

The proof of the following proposition is a standard computation:

Proposition 7.22 *For $\lambda \in \mathbb{C}^*$ and $\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigotimes^r A^\bullet$ holds:*

$$\lambda^N\left(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}\right) = (\lambda^N \varphi_{j_1}) \otimes \cdots \otimes (\lambda^N \varphi_{j_r}) \quad \textit{and}$$
$$\lambda^M\left(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r}\right) = (\lambda^M \varphi_{j_1}) \otimes \cdots \otimes (\lambda^M \varphi_{j_r}).$$

□

Both, N and M in (7.7) induce maps $N, M : H^0\bar{B}_s(A^\bullet, a) \to H^0\bar{B}_s(A^\bullet, a)$. The complex vector space $H^0\bar{B}_s(A^\bullet, a)$ does neither depend on $\vec{v}$ nor on $\vec{w}$. Therefore, we can study, how the lattice

$$\left(J_{\vec{v},\vec{w}}/J^{s+1}_{\vec{v},\vec{w}}\right)^*_{\mathbf{Z}} \subset H^0\bar{B}_s(A^\bullet, a)$$

behaves, when we move $\vec{v} \in (T_0\Delta)^*$ and $\vec{w} \in (T_{p_0}D_0)^*$. Let T and S be the monodromies of the local systems

$$\left\{\left(J_{\vec{v},\vec{w}}/J^{s+1}_{\vec{v},\vec{w}}\right)^*_{\mathbf{Z}}\right\}_{\vec{v}\in(T_0\Delta)^*} \text{ and } \left\{\left(J_{\vec{v},\vec{w}}/J^{s+1}_{\vec{v},\vec{w}}\right)^*_{\mathbf{Z}}\right\}_{\vec{w}\in(T_{p_0}D_0)^*}$$

respectively. The main result of this section is the following theorem.

Theorem 7.23 *$e^{-2\pi i N} = T$ and $e^{2\pi i M} = S$. For λ, $\mu \in \mathbb{C}^*$ holds:*

$$\left(J_{\lambda\vec{v},\mu\vec{w}}/J^{s+1}_{\lambda\vec{v},\mu\vec{w}}\right)^*_{\mathbf{Z}} = \lambda^{-N}\mu^M\left(J_{\vec{v},\vec{w}}/J^{s+1}_{\vec{v},\vec{w}}\right)^*_{\mathbf{Z}} \subset H^0\bar{B}_s(A^\bullet, a).$$

Proof: Since M and N commute, we may prove the theorem by proving the following two equations separately:

(i) $\left(J_{\lambda\vec{v},\vec{w}}/J^{s+1}_{\lambda\vec{v},\vec{w}}\right)^*_{\mathbf{Z}} = \lambda^{-N}\left(J_{\vec{v},\vec{w}}/J^{s+1}_{\vec{v},\vec{w}}\right)^*_{\mathbf{Z}} \subset H^0\bar{B}_s(A^\bullet, a)$,

(ii) $\left(J_{\vec{v},\mu\vec{w}}/J^{s+1}_{\vec{v},\mu\vec{w}}\right)^*_{\mathbf{Z}} = \mu^M\left(J_{\vec{v},\vec{w}}/J^{s+1}_{\vec{v},\vec{w}}\right)^*_{\mathbf{Z}} \subset H^0\bar{B}_s(A^\bullet, a)$.

We will prove (i) and leave it to the reader to prove (ii).

Let $\lambda : [0;1] \to \mathbb{C} \setminus \mathbb{R}^{\leq 0}$ be a path with $\lambda(0) = 1$. Let λ_0 be a positive real number such that $\lambda_0 < |\lambda(\varsigma)|$ for all $\varsigma \in [0;1]$.

Like always, we fix coordinates $(x, y) : W_{kl} = U^k_{kl} \times U^l_{kl} \to \mathbb{C}^2$ with respect to $\vec{v}$, that means $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle = \vec{v}$. For an appropriate $\rho > 0$ let $\tilde{y}_\varsigma = \tilde{y} : U^l_{kl} \to \mathbb{C}$ be another coordinate that depends continuously on $\lambda(\varsigma)$ resp. ς and that satisfies:

$$\begin{aligned} &\tilde{y}_\varsigma = y && \text{for } |y| \geq \frac{\rho}{\lambda_0}, \\ &\tilde{y}_\varsigma = \frac{1}{\lambda(\varsigma)}y && \text{for } |y| \leq \rho \text{ and} \\ &\tilde{y}_0 = y && \text{for all } y. \end{aligned}$$

For example, take a smoothing of the map that is given in polar coordinates by:

$$\begin{pmatrix}\tilde{r}\\ \tilde{\varphi}\end{pmatrix} = \begin{pmatrix}(r-\rho)\frac{|\lambda|-\lambda_0}{\lambda-|\lambda|\lambda_0} + \frac{\rho}{|\lambda|}\\ \varphi + \arg(\lambda)\left(\frac{\lambda_0 r-\rho}{\rho-\rho\lambda_0}\right)\end{pmatrix} \quad \text{for } \rho \leq r \leq \frac{\rho}{\lambda_0}.$$

Observe that for the coordinates $(x, \tilde{y}_\varsigma)$ holds: $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial \tilde{y}_\varsigma}\rangle = \lambda(\varsigma)\vec{v}$.

These coordinate changes $\tilde{y}_\varsigma^{-1} \circ y$ at any U^l_{kl} can be put together to an isotopy

$$\phi_\varsigma : Z_o \to Z_0,$$

which is defined to be $\tilde{y}_\varsigma^{-1} \circ y$ on any U^l_{kl} and the identity elsewhere. Note that $\phi_0 = id$ and $\phi_\varsigma(y) = \lambda(\varsigma)y$ for $y \in U^l_{kl}$ with $|y| \leq \rho$. Given a path $\gamma : [a;b] \to Z_0$ over $\vec{v}$ based at $\vec{w}$, we can define a homotopy H by $H(t,\varsigma) := \phi_\varsigma \circ \gamma(t)$. This $H(\cdot,\varsigma)$ is a path over $\lambda(\varsigma)\vec{v}$ based at $\vec{w}$.

Now take a Chen-closed $I = \sum_J a_J \; \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^{s} \bigotimes^r A^1$. We obtain by the definition of iterated integrals along paths over $\lambda(\varsigma)\vec{v}$ based at $\vec{w}$:

$$\int_{H(,\varsigma)} I = \int_\gamma \phi_\varsigma^* I. \tag{7.8}$$

Here the dga-automorphism $\phi_\varsigma^* : A^\bullet \to A^\bullet$ is defined by

$$(\phi_\varsigma|_{D_i})^* : E^\bullet(D_i \log P_i) \to E^\bullet(D_i \log P_i) \text{ and}$$

$$id : E^\bullet(\Delta^1) \otimes \Lambda^\bullet \to E^\bullet(\Delta^1) \otimes \Lambda^\bullet.$$

ϕ_ς^* induces an automorphism $\bigoplus_{r=1}^{s} \bigotimes^r \phi_\varsigma^*$ of $\bigoplus_{r=1}^{s} \bigotimes^r A^\bullet$, which we also denote by ϕ_ς^*. This is the ϕ_ς^* in (7.8).

We are done with the proof of the theorem if we can show for all paths $\gamma \in \pi_1(Z_{\vec{v}}, \vec{w})$:

$$\int_\gamma \phi_\varsigma^* I = \int_\gamma \lambda(\varsigma)^N I.$$

Also this statement will be proved by first looking at the local and then at the global situation.

Local Situation: Assume here that $\vartheta : [a; b] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$ is a path over $\vec{v}$ that meets the double point p_{kl} once with parameter value τ_0, where it changes from D_k to D_l. (Since $\phi_\varsigma^* I$ and $\lambda(\varsigma)^N I$ are nearby homotopy functionals, we do not have to worry about paths that *collide with a double point.*) Suppose the endpoint $\vartheta(b)$ of ϑ be such that $|y(\vartheta(b))| > \frac{\rho}{\lambda_0}$.

Notice that the two coordinate systems (x, y) and $(x, \tilde{y}_\varsigma)$ around p_{kl} give two different *localizations in a sector at the double point* p_{kl}

$$A^\bullet \to A_{kl}^\bullet \qquad \text{and} \qquad A^\bullet \to \tilde{A}_{kl}^\bullet$$

respectively. Denote the images of I under these natural maps by I_{kl} resp. $\tilde{I}_{kl}$. Observe moreover that ϕ_ς induces

$$\phi_\varsigma : U_{kl}^{k-} = \{y \notin \mathbb{R}^{\leq 0}\} \xrightarrow{\approx} \tilde{U}_{kl}^{k-} = \{\tilde{y}_\varsigma \notin \mathbb{R}^{\leq 0}\}.$$

Then we can extend the dga-automorphisms ϕ_ς^* and $\lambda(\varsigma)^N$ of $A^\bullet$ to dga-isomorphisms with the same names, ϕ_ς^* and $\lambda(\varsigma)^N$, from $\tilde{A}_{kl}^\bullet$ to $A_{kl}^\bullet$ like indicated in the following two commutative diagrams.

$$\begin{array}{ccccc} A^\bullet & \longrightarrow & \tilde{A}_{kl}^\bullet & \log \tilde{y} & w \\ {\scriptstyle\phi_\varsigma^*}\downarrow & & \downarrow{\scriptstyle\phi_\varsigma^*} & \downarrow & \downarrow \\ A^\bullet & \longrightarrow & A_{kl}^\bullet & \log \tilde{y} \circ \phi_\varsigma(y) & w, \end{array}$$

I. e. ϕ_ς on $\tilde{A}_{kl}^\bullet$ is the pull-back with ϕ_ς^*. Extend $\lambda(\varsigma)^N$ by

$$\begin{array}{ccccc} A^\bullet & \longrightarrow & \tilde{A}_{kl}^\bullet & \log \tilde{y} & w \\ {\scriptstyle\lambda(\varsigma)^N}\downarrow & & \downarrow{\scriptstyle\lambda(\varsigma)^N} & \downarrow & \downarrow \\ A^\bullet & \longrightarrow & A_{kl}^\bullet & \log \tilde{y}(y) & (w - \log \lambda). \end{array}$$

We know by Theorem 7.7 that for $\tilde{I}_{kl}$ and the points $\vartheta(b)$, $\vartheta(a)$ there is a function $\tilde{f}_{kl} \in \tilde{A}^0_{kl}$ such that holds:

$$\int_\vartheta I = \int_\vartheta \tilde{I}_{kl} = \tilde{f}_{kl}(\tilde{y} \circ \vartheta(b)) - \tilde{f}_{kl}(\tilde{y} \circ \vartheta(a)).$$

Since both, ϕ_ς^* and $\lambda(\varsigma)^N$, are isomorphisms of dga's, which preserve the values of functions on $\vartheta(b)$ and $\vartheta(a)$, the pull-back $\phi_\varsigma^* \tilde{f}_{kl}$ is such a function for $\phi_\varsigma^* \tilde{I}_{kl}$ and $\lambda(\varsigma)^N \tilde{f}_{kl}$ is such a function for $\lambda(\varsigma)^N \tilde{I}_{kl}$. That is to say:

$$\int_\vartheta \phi_\varsigma^* I = \int_\vartheta \phi_\varsigma^* \tilde{I}_{kl} = \left(\phi_\varsigma^* \tilde{f}_{kl}\right)(\tilde{y} \circ \vartheta(b)) - \left(\phi_\varsigma^* \tilde{f}_{kl}\right)(\tilde{y} \circ \vartheta(a))$$

and

$$\int_\vartheta \lambda(\varsigma)^N I = \int_\vartheta \lambda(\varsigma)^N \tilde{I}_{kl} = \left(\lambda(\varsigma)^N \tilde{f}_{kl}\right)(\tilde{y} \circ \vartheta(b)) - \left(\lambda(\varsigma)^N \tilde{f}_{kl}\right)(\tilde{y} \circ \vartheta(a)).$$

If $\tilde{f}_{kl}$ is given by

$$\tilde{f}_{kl} = \sum_{\nu=0}^{m} g_{k,\nu}(x) \log^\nu x + \sum_{\nu=0}^{m} g_{l,\nu}(\tilde{y}) \log^\nu \tilde{y} + P(\xi, v, w)$$

then

$$\phi_\varsigma^* \tilde{f}_{kl} = \sum_{\nu=0}^{m} g_{k,\nu}(x) \log^\nu x + \sum_{\nu=0}^{m} g_{l,\nu}(y) \log^\nu y + P(\xi, v, w)$$

and

$$\lambda(\varsigma)^N \tilde{f}_{kl} = \sum_{\nu=0}^{m} g_{k,\nu}(x) \log^\nu x + \sum_{\nu=0}^{m} g_{l,\nu}(\tilde{y}(y)) \log^\nu \left(\tilde{y}(y)\right) + P(\xi, v, w - \log \lambda).$$

Because of $\tilde{y}\big(y \circ \vartheta(b)\big) = y \circ \vartheta(b)$ as $|y \circ \vartheta(b)| > \frac{\rho}{\lambda_0}$, we finally find:

$$\int_\vartheta \phi_\varsigma^* I = \int_\vartheta \lambda(\varsigma)^N I.$$

Global Situation: Let $\gamma : [a; b] \to Z_0$ be a path over $\vec{v}$ based at $\vec{w}$ that meets the set of double points $\{p_{kl}\}$ with parameter values $a = \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1} = b$. At the double point $p_0 \in D_0$, we have the embedding $x = p : D_0 \to \mathbb{C}$. Like before, let σ be the straight path $t \mapsto (1-t)$ from 1 to $0 \in D_0$. Decompose $\sigma \star \gamma \star \sigma^{-1} = \gamma_0 \star \cdots \star \gamma_{m+1}$ such that for each $\gamma_\kappa : [a_\kappa; b_\kappa] \to Z_0$, the intervall $[a_\kappa; b_\kappa]$ contains only the parameter value τ_κ in its interior. Moreover, there is an $\varepsilon_0 > 0$ with which we can decompose furthermore each of the paths γ_κ into

$\gamma_\kappa = \alpha_\kappa \star \gamma_\kappa \star \beta_\kappa$ such that α_κ resp. β_κ lie entirely in one component and each δ_κ is a map

$$\delta_\kappa : [\tau_\kappa - \varepsilon_0; \tau_\kappa + \varepsilon_0] \to U_{kl}^{k-} \cup \{p_{kl}\} \cup U_{kl}^{l-}$$

for some $[k < l]$.

We know that if ϑ is one of the paths α_κ, γ_κ, β_κ then $\int_\vartheta \phi_\varsigma^* I = \int_\vartheta \lambda(\varsigma)^N I$. It remains to show that if ϑ and ϑ', where $\vartheta \star \vartheta'$ is a connected path, have this property then also $\vartheta \star \vartheta'$ does: $\int_{\vartheta\star\vartheta'} \phi_\varsigma^* I = \int_{\vartheta\star\vartheta'} \lambda(\varsigma)^N I$. But this follows from the proof of Lemma 7.12 (the choice of the $I_1^{pq}, \dots I_M^{pq}$ is compatible with ϕ_ς^* and $\lambda(\varsigma)^N$). That proves the theorem. □

We obtain as a consequence of Theorem 7.23

Theorem 7.24 *The family of mixed Hodge structures*

$$\left\{ \left(J_{\vec{v},\vec{w}} / J_{\vec{v},\vec{w}}^{s+1} \right)^* \right\}_{\vec{v} \in (T_0\Delta)^*}$$

is a nilpotent orbit of mixed Hodge structure. □

7.6 The MHS on the Fundamental Group of the Central Fiber

Let $\pi_1(Z_0, p_0)$ be the fundamental group of the central fiber and let $\mathbb{Z}\pi_1(Z_0, p_0)$ be its group ring with augmentation ideal J_0. In this section we define a MHS on the fundamental group of the central fiber. Apart from the contribution due to the non compact disk D_0, the construction of this MHS is a special case of the general construction of a MHS on the fundamental group of a complex algebraic varietiy as introduced by Hain [Hai87a]. The main result here in this section is that the obvious group homomorphism

$$c : \pi_1(Z_{\vec{v}}, \vec{w}) \to \pi_1(Z_0, p_0)$$

induces an inclusion of MHSs:

$$c^* : \left(J_0 / J_0^{s+1}\right)^* \to \left(\vec{J} / \vec{J}^{s+1}\right)^*.$$

Most proofs are very similar to the proofs of the corresponding statements for $\pi_1(Z_{\vec{v}}, \vec{w})$ and are therefore left to the reader.

7.6.1 A DGA of Differential Forms on the Central Fiber

Now let us construct a dga $A_0^\bullet$ which computes the cohomology of Z_0 and allows to define the MHS on $\pi_1(Z_0, p_0)$.

Define

$$B_0^\bullet := \mathbb{C} \oplus \bigoplus_{i>0} E^\bullet(D_i) \oplus \bigoplus_{[k<l]} E^\bullet(p_{kl}) \otimes E^\bullet(\Delta^1).$$

By setting $\mathbb{C} = \bigwedge^\bullet(\frac{dp}{p})$ and $E^\bullet(p_{kl}) = \Lambda^0$ we may consider $B_0^\bullet$ as sub-dga of the complex $B^\bullet$ as it was defined on page 90. Now define the dga:[4]

$$A_0^\bullet := B_0^\bullet \cap A^\bullet.$$

Note that

$$A_0^\bullet = W_0 A^\bullet,$$

i. e. the dga $A_0^\bullet$ consists of exactly those elements in $A^\bullet$, which have no poles and no "$\log t = u$" 's. $A_0^\bullet$ inherits a Hodge- and a weight filtration $F^\bullet$ and $W_\bullet$ as well as an augmentation map $a : A_0^\bullet \to \mathbb{C}$ from $A^\bullet$.

Similarly to Theorem 6.2 on page 92 one can prove:

Theorem 7.25 $H^\bullet(A_0^\bullet) \cong H^\bullet(Z_0; \mathbb{C})$. □

7.6.2 Collapsing the Vanishing Cycles

A path $\gamma : [0;1] \to Z_0$ over $\vec{v}$ based at $\vec{w}$ is in particular a path in D^+ based at p_0. We leave it to the reader to prove:

Proposition 7.26 *The obvious map*

$$c : \pi_1(Z_{\vec{v}}, \vec{w}) \to \pi_1(Z_0, p_0).$$

is a well-defined surjective group homomorphism.

7.6.3 Iterated Integrals on the Central Fiber

Since $A_0^\bullet$ is a sub-dga of $A^\bullet$ we may define:

Definition 7.27 *Let* $I = \sum_{|J| \le s} a_J \; \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_r} \in \bigoplus_{r=1}^s \bigotimes^r A_0^1$ *be Chen-closed and let* $[\gamma] \in \pi_1(Z_0, p_0)$ *be such that it is represented by a path* γ *over* $\vec{v}$ *based at* $\vec{w}$. *Then define*

$$\int_{[c(\gamma)]} I := \int_\gamma I.$$

Now let J_0 be the augmentation ideal in the group ring $\mathbb{Z}\pi_1(Z_0, p_0)$. Similarly to Theorem 6.8 on page 99 one proves:

Theorem 7.28 *The integration of closed forms in* A_0^1 *along elements of* $\pi_1(Z_0, p_0)$ *defines an isomorphism*

$$H^1(A_0^\bullet) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}\left(J_0/J_0^2; \mathbb{C}\right).$$

[4] We avoid to introduce the language of simplicial and cosimplicial objects here. However, the complex $A_0^\bullet$ is isomorphic to the de Rham complex of the (complex part of the) cosimplicial mixed Hodge complex on the simplicial variety Z_0.

Since the integration maps are by definition compatible with $c : \pi_1(Z_{\vec{v}},\vec{w}) \to \pi_1(Z_0,p_0)$, we have an inclusion

$$H^1(A_0^\bullet) \hookrightarrow H^1(A^\bullet).$$

Define for $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$

$$H^1(A_0^\bullet)_R := H^1(A^\bullet)_R \cap H^1(A_0^\bullet) \text{ and } \left(J_0/J_0^{s+1}\right)_R^* := \operatorname{Hom}_{\mathbb{Z}}\left(J_0/J_0^{s+1};R\right).$$

It follows from the proof of Theorem 6.11 on page 101 that holds:

Theorem 7.29

$$(J_0/J_0^2)^* := \left(H^1(A_0^\bullet)_{\mathbb{Z}},\ (H^1(A_0^\bullet)_{\mathbb{Q}},W_\bullet),\ (H^1(A_0^\bullet),W_\bullet,F^\bullet)\right) = W_1(\vec{J}/\vec{J}^2)^*$$

In particular, $(J_0/J_0^2)^*$ is a $\mathbb{Z}$-MHS of possible weights 0 and 1.

Moreover, one can prove (similarly to Theorem 7.17 on page 136):

Theorem 7.30 *Integration of Chen-closed elements in $\bigoplus_{r=1}^s \bigotimes^s A_0^1$ along elements of $\pi_1(Z_0,p_0)$ defines an isomorphism of complex vector spaces*

$$H^0\bar{B}_s(A_0^\bullet,a) \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}}\left(J_0/J_0^{s+1};\mathbb{C}\right).$$

□

Since the integration maps are compatible with c and since $\vec{J}/\vec{J}^{s+1} \to J_0/J_0^{s+1}$ is surjective, we find

Corollary 7.31 $H^0\bar{B}_s(A_0^\bullet,a) \hookrightarrow H^0\bar{B}_s(A^\bullet,a)$. □

Similarly to Theorem 7.20 on page 139 one proves:

Theorem 7.32 *For any $s \geq 1$ is*

$$\left(J_0/J_0^{s+1}\right)^* := \left(\left(J_0/J_0^{s+1}\right)_{\mathbb{Z}}^*,\ \left(\left(J_0/J_0^{s+1}\right)_{\mathbb{Q}}^*,W_\bullet\right)\ \left(\left(J_0/J_0^{s+1}\right)_{\mathbb{C}}^*,W_\bullet,F^\bullet\right)\right)$$

a mixed Hodge structure and the short exact sequence

$$0 \to \left(J_0/J_0^s\right)^* \to \left(J_0/J_0^{s+1}\right)^* \to H^1(A_0^\bullet)^{\otimes s} \to 0 \tag{7.9}$$

is a short exact sequence of MHSs.

Finally, let us conclude this section by stating the following easy to prove theorem.

Theorem 7.33 *The map $c : \pi_1(Z_{\vec{v}},\vec{w}) \to \pi_1(Z_0,p_0)$ induces an inclusion of MHSs*

$$\left(J_0/J_0^{s+1}\right)^* \hookrightarrow \left(\vec{J}/\vec{J}^{s+1}\right)^*.$$

□

Chapter 8

Preparation of the Curve Singularity Case

In this last chapter, we finally want to show, how the setting of 5.2 arises in the case of irreducible plane curve singularities. Throughout the whole chapter let

$$f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$$

be an irreducible plane curve singularity (IPCS) with Puiseux pairs $(m_1, n_1), \ldots, (m_g, n_g)$.

First, we will construct a map $h : (Z, D^+) \to (\Delta, 0)$ with the properties that we described earlier (cf. 5.2) and that is related to f in a natural way, and secondly, we are going to construct tangent vectors in $(T_0\Delta)^*$ and $(T_{p_0}D_0)^*$ in a natural way.

The construction of this map $h : (Z, D^+) \to (\Delta, 0)$ will be done by a construction, which is described in Steenbrink, [Ste77]. We refer to this process as *semistable reduction.*

In other words, we consider the embedded resolution (see p. 148) $\pi : (\tilde{X}, E^+) \to (\mathbb{C}^2, 0)$ of f. We define the integer d to be the lowest common multiple of all multiplicities of $(f \circ \pi)^{-1}(0)$ in the resolution. Let E_0 be the strict preimage in the embedded resolution. We construct a map $h' : (Z', D'^+) \to (\Delta, 0)$ by performing first a basechange for the map $f \circ \pi : (\tilde{X}, E^+) \to (\mathbb{C}, 0)$ with the map $\Delta \to \mathbb{C}$, that maps t to t^d. The space germ (Z', D'^+) is then the normalization of this space germ obtained by this basechange. This (Z', D'^+) might still have cyclic quotient singularities, which can be resolved by inserting chains of $\mathbb{P}^1$'s. Then we obtain $h : (Z, D^+) \to (\Delta, 0)$. The given map from Z to $\tilde{X}$ induces an isomorphism $T_{p_0}D_0 \cong T_{p_0}E_0$ with which we identify these two tangent spaces.

The tangent spaces $T_0\Delta$ and $T_{p_0}D_0$ resp. $T_{p_0}E_0$, are 1-dimensional complex vector spaces. Therefore $(T_0\Delta)^*$, $(T_{p_0}D_0)^*$ and $(T_{p_0}E_0)^*$ carry natural $\mathbb{C}^*$ actions. We will construct a natural orbit $\mathcal{S}(\frac{\partial}{\partial t})$ of d tangent vectors in $(T_0\Delta)^*$ under the induced action of the group μ_d of d-th roots of unity. These tangent vectors will play the role of the vector $\vec{v} \in (T_0\Delta)^*$ before.

Moreover, with the definitions $m := m_1 \cdots m_g$ and $k := n_1 m_2 \cdots m_g$, we will define a natural orbit of $m \cdot k$ tangent vectors in $(T_{p_0}E_0)^*$ under the induced action of the group $\mu_{m\cdot k}$ of $m \cdot k$-th roots of unity. We will refer to this orbit as to *the monstrance* of the IPCS and denote it by $\mathcal{M}(\frac{\partial}{\partial t})$. Its elements will serve as *base vectors* like the $\vec{w}$ in the discussion before.

Summarizing this little preview, we can say that, given $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, we will first construct an $h : (Z, D^+) \to (\Delta, 0)$ like in the setting 5.2 and a commutative diagram of space germs

$$\begin{array}{ccc} (Z, D^+) & \xrightarrow{P} & (\mathbb{C}^2, 0) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ (\Delta, 0) & \longrightarrow & (\mathbb{C}, 0) \\ t & \longmapsto & t^d\,, \end{array}$$

where P induces an isomorphism between $Z \setminus h^{-1}(0)$ and $(\mathbb{C}^2 \setminus f^{-1}(0)) \times_{\mathbb{C}^*} \Delta^*$.

Moreover we construct an orbit $\mathcal{S}(\frac{\partial}{\partial t}) = \{\zeta_d^\nu \vec{v}\}_{\nu=0,\dots,d-1}$ of the action of the d-th roots of unity on $(T_0\Delta)^*$ and an orbit $\mathcal{M}(\frac{\partial}{\partial t}) = \{\zeta_{mk}^\nu \vec{w}\}_{\nu=0,\dots,mk-1}$ of the action of the mk-th roots of unity on $(T_{p_0}D_0)^*$. The cyclic group μ_d acts on the fibration h and the monstrance $\mathcal{M}(\frac{\partial}{\partial t}) \subset T_{p_0}D_0$ is invariant under this action.

$(h,\ P,\ \{\zeta_d^\nu \vec{v}\}_{\nu=0,\dots,d-1},\ \{\zeta_{mk}^\nu \vec{w}\}_{\nu=0,\dots,mk-1})$ are defined in such a way that any right-equivalence between two IPCSs transforms these data into each other. That is, what the word *natural* refers to in this context.

Finally, we will define *mixed Hodge structure on the nearby fundamental group of an irreducible plane curve singularity* and will finish with an example.

8.1 Semistable Reduction

Consider the IPCS $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and let $\pi : (\tilde{X}, E^+) \to (\mathbb{C}^2, 0)$ be its embedded (minimal) resolution. Let E denote the divisor $(f \circ \pi)^{-1}(0)$ with components $E = \bigcup_{i \geq 0} E_i$, where E_i has multiplicity e_i. Let E_0 be *the strict preimage* of $f^{-1}(0)$ in $\tilde{X}$; that is the closure of $(f \circ \pi)^{-1}(0) \setminus \pi^{-1}(0)$ in $\tilde{X}$. E^+, *the exceptional divisor*, is defined to be $\bigcup_{i>0} E_i$. The E_i's for $i > 0$ are all rational curves. Call q_0 the intersection point $E_0 \cap E^+$ and note that $(E_0, q_0) \cong (\mathbb{C}, 0)$.

Let $d = \operatorname{lcm}_i e_i$ and let $\theta : \Delta \to \mathbb{C}$ be the map $t \mapsto t^d$. Now consider the normalization of the fibered product of $f \circ \pi$ and θ:

$$Z' := \widetilde{\tilde{X} \times_{\mathbb{C}} \Delta}.$$

There are projections $h' : Z' \to \Delta'$ and $P' : Z' \to \tilde{X}$. Define $D'^+ := (\pi \circ P')^{-1}(0)$ and let D'_0 be the closure of $(f \circ \pi \circ P')^{-1}(0) \setminus (\pi \circ P')^{-1}(0)$ in Z'.

Proposition 8.1 *Z' has exclusively cyclic quotient singularities. More precisely, for each singular point $p \in Z'$ there exists a natural number $n_p > 0$ and an isomorphism*

$$\varphi_p : \mathcal{O}_{Z',p} \xrightarrow{\cong} \frac{\mathbb{C}\{\alpha,\beta,\eta\}}{(\alpha\beta - \eta^{n_p})}$$

that maps h' to η.

Proof: Due to the fact that the normalization (cf. [KK83] §71, p. 302) can be constructed locally and by the way the embedded resolution is defined, it suffices to prove the assertion for a space Y' of the following type.

Let a, $b > 0$ be two natural numbers, which divide d. Define a' and b' by: $d = a\,a'$ and $d = b\,b'$. Then let $g : \mathbb{C}^2 \to \mathbb{C}$ be the function $g(u,v) = u^a v^b$. Recall that $\theta(t) = t^d$. We are going to study the normalization of the fibered product of g and θ:

$$Y' := \widetilde{\mathbb{C}^2 \times_{\mathbb{C}} \Delta}.$$

Let $P : \mathbb{C}^2 \times_{\mathbb{C}} \Delta \to \Delta$ be the projection to the second factor. $\mathbb{C}^2 \times_{\mathbb{C}} \Delta \subset \mathbb{C}^3$ is described by the equation $u^a v^b - t^d = 0$ and P maps (u,v,t) to t.

We want to investigate $\mathbb{C}^2 \times_{\mathbb{C}} \Delta$ in a point $p_0 = (u_0, v_0, t_0)$. Let us distinguish the following cases.

(i) $u_0 \neq 0$; $v_0 \neq 0$: In this case, $\mathbb{C}^2 \times_{\mathbb{C}} \Delta$ is smooth in (u_0, v_0, t_0): Define $x := u - u_0$ and $y := v - v_0$, then Y' is locally around p_0 given by the equation $(x + u_0)^a (y + v_0)^b - t^d$, which has a smooth zero set around $(x,y) = (0,0)$.

(ii) $u_0 = 0$; $v_0 \neq 0$: $\mathbb{C}^2 \times_{\mathbb{C}} \Delta$ is reducible in p_0 and decomposes into a nonsingular factors: Let again $y := v - v_0$. Then with $\zeta_a = e^{\frac{2\pi i}{a}}$ we find:

$$\begin{aligned} u^a v^b - t^d &= u^a (y + v_0)^b - t^d \\ &= \prod_{\nu=1}^{a} \left(u \sqrt[a]{(y + v_0)^b} - \zeta_a^\nu t^{a'} \right). \end{aligned}$$

The normalization just separates these non singular factors.

(iii) $u_0 = 0$; $v_0 = 0$: Let $m := \gcd(a,b)$ and define a'', b'', d'', n by $a = ma''$, $b = mb''$ and $d = md''$ (note that $(a'', b'') = 1$) as well as $d'' = a''b''n$. Then $\mathbb{C}^2 \times_{\mathbb{C}} \Delta$ decomposes into m (possibly singular) factors ($\zeta_m = e^{\frac{2\pi i}{m}}$):

$$\begin{aligned} u^a v^b - t^d &= \left(u^{a''} v^{b''}\right)^m - \left(t^{d''}\right)^m \\ &= \prod_{\nu=1}^{m} \left(u^{a''} v^{b''} - \zeta_m^\nu t^{d''} \right). \end{aligned}$$

The normalization Y' first separates these components into the disconnected sum of the normalizations of each of these components, which we describe now. Fix ν and let $\tilde{t} := e^{\frac{2\pi i\nu}{md''}}\, t$. The normalization of the component $\left(u^{a''}v^{b''} - \zeta_m^{\nu} t^{d''}\right) = \left(u^{a''}v^{b''} - \tilde{t}^{d''}\right)$ is given by

$$\begin{array}{ccc} \{\alpha\beta - \eta^n = 0\} & \longrightarrow & \left\{u^{a''}v^{b''} - \tilde{t}^{d''} = 0\right\} \\ (\alpha, \beta, \eta) & \longmapsto & (\alpha^{b''}, \beta^{a''}, \eta). \end{array}$$

It remains to check that this map is surjective and 1-1 off the singular locus. Recall $(a'', b'') = 1$ and $d'' = a''b''n$. Let $(u, v, \tilde{t})$ be such that $\tilde{t} \neq 0$ and $u^{a''}v^{b''} = \tilde{t}^{d''}$. Then let $\tilde{t}^n = \zeta_{a''}^r \zeta_{b''}^s \sqrt[b'']{u}\, \sqrt[a'']{v}$ for some r, s and fixed roots $\sqrt[b'']{u}$ and $\sqrt[a'']{v}$. We find the preimage of $(u, v, \tilde{t})$ under this map by $\eta = \tilde{t}$ and $\alpha = \zeta_{b''}^i \sqrt[b'']{u}$ as well as $\beta = \zeta_{a''}^j \sqrt[a'']{v}$. Because of $\alpha\beta = \eta^n$ and the fact that $(a'', b'') = 1$, the numbers $i \in \{0, \ldots, b'' - 1\}$ and $j \in \{0, \ldots, a'' - 1\}$ are uniquely determined. This proves the proposition. □

Remark 8.2 Since $e_0 = 1$, the above proof also shows that the map P' of Z' to the embedded resolution $\tilde{X}$ induces an isomorphism between D_0' and the strict preimage E_0 in $\tilde{X}$.

With the notation of 5.2 we are ready to prove now:

Theorem 8.3 *For an irreducible plane curve singularity $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ there is a map $h : (Z, D^+) \to (\Delta, 0)$ and a commutative diagram of space germs*

$$\begin{array}{ccc} (Z, D^+) & \xrightarrow{P} & (\mathbb{C}^2, 0) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ (\Delta, 0) & \longrightarrow & (\mathbb{C}, 0) \\ t & \longmapsto & t^d, \end{array}$$

such that P induces an isomorphism over Δ^* between the spaces $Z \setminus h^{-1}(0)$ and $(\mathbb{C}^2 \setminus f^{-1}(0)) \times_{\mathbb{C}^*} \Delta^*$.

Proof: What remains to show after that we proved Proposition 8.1 is the well-known fact that a singularity of the type: $\alpha\beta - \eta^n$ may be resolved by replacing the singular point by a chain of $(n-1)$ copies of $\mathbb{P}^1$. This is done as follows (cf. for instance [BK91]).

Consider n copies of $\mathbb{C}^2$ with coordinates (u_i, v_i) for $i = 1, \ldots, n$. These spaces are glued together by the following glueing maps:

$$\begin{array}{llcl} \varphi_i : & \mathbb{C}^2 \setminus \{v_i = 0\} & \to & \mathbb{C}^2 \setminus \{u_{i+1} = 0\} \\ & (u_i, v_i) & \mapsto & (\frac{1}{v_i},\, u_i v_i^2) = (u_{i+1}, v_{i+1}). \end{array}$$

Call the result M and observe that M is smooth. Now the resolution of the variety $\{\alpha\beta - \eta^n = 0\}$ is given by the map $M \to \{\alpha\beta - \eta^n = 0\}$, that is defined on the coordinates (u_i, v_i) by

$$(u_i, v_i) \mapsto \left(u_i^i v_i^{i-1};\ u_i^{n-i} v_i^{n+1-i};\ u_i v_i\right) = (\alpha, \beta, \eta).$$

Notice that this map is well-defined. In this way we can resolve all the singularities of Z' and find the map $h : (Z, D^+) \to (\Delta, 0)$ with the desired properties. $\square$

8.2 Canonical Tangent Vectors

In this section we first recall some well-known facts about the map $t \mapsto t^d$. These observations will be used to construct an orbit under the action of the d-th roots of unity on $(T_0\Delta)^*$.

A careful study of the Puiseux expansion will show subsequently that there is a natural orbit under the action of the mk-th roots of unity on $(T_{p_0}D_0)^*$, which we call *the monstrance* of f.

8.2.1 Pulling Back Tangent Vectors

Let Δ and Δ' be two disks in $\mathbb{C}$ around 0. Always, when we are given a holomorphic map of multiplicity m, say

$$\varphi : \Delta \to \Delta',$$

then the differential of φ in 0 is zero, when $m > 1$. However, a tangent vector $\vec{v}\,' \in (T_0\Delta')^*$ defines in a natural way m tangent vectors $\zeta_m^\nu \vec{v} \in (T_0\Delta)^*$ for $\nu = 0, \dots, m-1$.

Let us refer to the set $\left\{\zeta_m^\nu \vec{v} \in (T_0\Delta)^* \middle| \, \nu = 0, \dots, m-1\right\}$ as to *the inverse star of $\vec{v}\,'$ under φ*. We will give two descriptions of this phenomenon: a concrete one and a *fancy* one.

The concrete description is as follows. Choose a coordinate z' on $(\Delta', 0)$ such that $\vec{v}\,' = \frac{\partial}{\partial z'}$. Then there is up to a choice of an m-th root of unity only one coordinate z on Δ such that: $z' \circ \varphi = z^m$. Define the m tangent vectors by $\zeta_m^\nu \frac{\partial}{\partial z} \in (T_0\Delta)^*$, where $\nu = 0, \dots, m-1$. Notice that this definition does not depend on the choice of z'.

The abstract description is the following. Let $\mathfrak{m}$ resp. $\mathfrak{m}'$ be the maximal ideal in $\mathcal{O}_{\Delta,0}$ resp. $\mathcal{O}_{\Delta',0}$ and let $\mathfrak{m}/\mathfrak{m}^2$ resp. $\mathfrak{m}'/\mathfrak{m}'^2$ be the corresponding Zariski cotangent spaces. Observe that by the duality

$$\begin{array}{ccc} T_0\Delta \otimes (\mathfrak{m}/\mathfrak{m}^2) & \to & \mathbb{C} \\ (X \otimes g) & \mapsto & X(g)(0) \end{array}$$

a choice of a tangent vector in $T_0\Delta$ corresponds to the choice of an element in $\mathfrak{m}/\mathfrak{m}^2$. Now φ induces a map

$$\begin{array}{cccc} \varphi^* : & \mathfrak{m}'/\mathfrak{m}'^2 & \to & \mathfrak{m}^m/\mathfrak{m}^{m+1} \\ & g' & \mapsto & g' \circ \varphi. \end{array}$$

On the other hand we have the composition of natural maps:

$$\begin{array}{ccccc} \mathfrak{m}/\mathfrak{m}^2 & \to & (\mathfrak{m}/\mathfrak{m}^2)^{\otimes m} & \to & \mathfrak{m}^m/\mathfrak{m}^{m+1} \\ g & \mapsto & g \otimes \cdots \otimes g & & \\ & & a_1 \otimes \cdots \otimes a_m & \mapsto & a_1 \cdots a_m. \end{array}$$

Each non-zero element in $\mathfrak{m}^m/\mathfrak{m}^{m+1}$ has m inverse images in $\mathfrak{m}/\mathfrak{m}^2$ under this composition of maps. This shows that a tanget vector $\vec{v}\,' \in (T_0\Delta')^*$ determines m tangent vectors in $(T_0\Delta)^*$ in a natural way.

For irreducible plane curve singularities $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, the tangent vector $\frac{\partial}{\partial t} \in (T_0\mathbb{C})^*$ is preserved under right-equivalences. Denote by

$$\mathcal{S}(\frac{\partial}{\partial t}) = \{\zeta_d^\nu \vec{v} \in (T_0\Delta)^* | \nu = 0, \ldots, d-1\}$$

the inverse star of $\frac{\partial}{\partial t}$ under the map $\theta : t \mapsto t^d$.

8.2.2 The Monstrance of f

The next thing to do, is to show there is a natural orbit under the action of the mk-th roots of unity on $(T_{p_0}E_0)^*$, the tangent space at the strict transform in the resolution. By virtue of Remark 8.2 this immediately gives tangent vectors in $T_{q_0}D_0$, which are invariant under the action of μ_d on h.

The idea for the construction is the following. We know that if we have a parametrization $\tau : (\mathbb{C}, 0) \to (E_0, q_0)$ of the strict preimage, then this defines a tangent vector in $T_{q_0}E_0$, namely: $\frac{\partial}{\partial \tau}$. The Puiseux expansion is a parametrization $p : (\mathbb{C}, 0) \to (f^{-1}(0), 0)$ with the property that it can be lifted to a smooth parametrization $\tilde{p} : (\mathbb{C}, 0) \to (E_0, q_0)$. In this way, a Puiseux expansion yields a tangent vector in $T_{q_0}E_0$. Now given an IPCS f, we can always find coordinates (x, y) of $\mathbb{C}^2$ such that

$$f(x, y) = 1 \cdot x^m + \cdots + 1 \cdot y^k + \cdots ,$$

where x^m and y^k are the smallest pure powers of x and y respectively. Up to first order, there are $m \cdot k$ such coordinate systems. This requirement on the coordinates with respect to f determines the Puiseux expansions up to first order. Up to first order, we get $m \cdot k$ canonical parametrizations of (E_0, q_0) and therefore $m \cdot k$ canonical tangent vectors.

But let us give a geometric meaning to this construction. Since we do not want to fix coordinates on $\mathbb{C}^2$ for the moment, we let $(X, x_0) := (\mathbb{C}^2, 0)$.

Definition 8.4 Let $\iota : (L, x_0) \hookrightarrow (X, x_0)$ be a 1-dimensional smooth subspace of (X, x_0). We will call a map

$$\mathbf{P} : (X, x_0) \to (L, x_0)$$

a projection of (X, x_0) onto (L, x_0), if the induced map

$$T_{x_0}L \xrightarrow{D\iota} T_{x_0}X \xrightarrow{D\mathbf{P}} T_{x_0}L$$

is the identity.

Now let $\iota : (R, x_0) \hookrightarrow (X, x_0)$ be the tangent to $(f^{-1}(0), x_0) \subset (X, x_0)$. We formulate the following theorem.

Theorem 8.5 *Let $f : (X, x_0) \to (\mathbb{C}, 0)$ be an irreducible plane curve singularity with Puiseux pairs $(m_1, n_1), \dots, (m_g, n_g)$. Set $m := m_1 \cdots m_g$ and $k := n_1 m_2 \cdots m_g$.*

Let $\iota : (R, x_0) \hookrightarrow (X, x_0)$ be the tangent and $\pi : (\tilde{X}, E^+) \to (X, x_0)$ be the embedded resolution of f. Then for any projection $\mathbf{P} : (X, x_0) \to (R, x_0)$, the composition ϕ of maps

$$(E_0, q_0) \xrightarrow{\pi_{|(E_0,q_0)}} (X, x_0) \xrightarrow{\mathbf{P}} (R, x_0) \xrightarrow{f_{|(R,x_0)}} (\Delta, 0)$$

has multiplicity $m \cdot k$ and the inverse star of the tangent vector $\frac{\partial}{\partial t}$ under ϕ does not depend on the choice of the projection $\mathbf{P}$.

Proof: Choose coordinates (x, y) on (X, x_0) such that the following two conditions are satisfied:

- The tangent line (R, x_0) of $(f^{-1}(0), x_0)$ is given by $x = 0$.
- $f(x, y) = 1 \cdot x^m + \cdots + 1 \cdot y^k + \cdots$, where x^m and y^k are the smallest pure powers of x and y respectively.

Then the Puiseux expansion of f with respect to (x, y) has the following form, where $m < k$ (f is irreducible):

$$\begin{aligned} x(\tau) &= a_k \tau^k + a_{k+1} \tau^{k+1} + \cdots \\ y(\tau) &= \tau^m \end{aligned}$$

and $a_k^m = -1$. Moreover we can find f back by:

$$f(x, y) = \prod_{\nu=0}^{m-1} \left(x - x(\zeta_m^\nu y^{\frac{1}{m}}) \right).$$

τ can be used as a coordinate on the strict transform (E, q_0). We may take $\tilde{y} := y_{|(R, x_0)}$ as coordinate on the tangent line (R, x_0).

Any projection $\mathbf{P} : (X, x_0) \to (R, x_0)$ is a priori given by $\tilde{y} \circ \mathbf{P}(x, y) = y(\mathbf{P}(x, y)) = Ax + By + \mathbf{h}(x, y)$ and $\mathbf{h}(x, y) \in \mathfrak{m}^2_{(X, x_0)}$. The condition $D\mathbf{P}_{|T_{x_0} R} = id$ means here: $B = 1$. Hence, we find for a unit $u(\tau) \in \mathbb{C}\{\tau\} \setminus \mathfrak{m}$ with $u(0) = \zeta_m^\nu$ (the irreducibility of f implies $m < k$):

$$\begin{aligned} \tilde{y} \circ \mathbf{P}\,(x(\tau), y(\tau)) &= \tau^m + \tau^{m+1} (\cdots) \\ &= \tau^m \cdot u(\tau)^m. \end{aligned}$$

Therefore

$$\begin{aligned} \phi(\tau) &= f\,(0, \tilde{y} \circ \mathbf{P}\,(x(\tau),\ y(\tau))) \\ &= (-1)^m \prod_{\nu=0}^{m-1} x\,(\zeta_m^\nu \tau \cdot u(\tau)) \\ &= (-1)^m \prod_{\nu=0}^{m-1} \left(a_k (\zeta_m^\nu \tau \cdot u)^k + a_{k+1} (\zeta_m^\nu \tau \cdot u)^{k+1} + \cdots \right) \\ &= (-1)^{m+k} a_k^m \tau^{m \cdot k} + \tau^{m \cdot k + 1} (\cdots). \end{aligned}$$

This shows that $\phi(\tau)$ has multiplicity $m \cdot k$ and the term of lowest order does not depend on the choice of $\mathbf{P}$. □

Definition 8.6 Define *the monstrance of f* to be the inverse star of $\frac{\partial}{\partial t}$ under a map ϕ like in Theorem 8.5 and denote it by

$$\mathcal{M}(\frac{\partial}{\partial t}) = \{\zeta_{mk}^{\nu}\vec{w} \in (T_{q_0}E_0)^* | \nu = 0, \dots, mk-1\}$$

Remark 8.7 $m \cdot k$ does not divide d in general. For instance in the example of Section 8.4 is $m = 4$, $k = 6$ and $d = 156$.

However, if f has only one Puiseux pair $(m, k) = (m_1, n_1)$, then $m \cdot k$ divides d as one easily sees by blowing up twice.

8.3 The Invariant

Here finally we are going to associate to an irreducible plane curve singularity an invariant of its right-equivalence class. In fact it is an invariant of the following (apparently coarser) equivalence relation.

Definition 8.8 We say that two plane curve singularities $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are *right-equivalent to first order* if there are isomorphisms $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$g \circ \phi = \varphi \circ f \quad \text{and} \quad \varphi_* : T_0\mathbb{C} \to T_0\mathbb{C} \text{ is the identity.}$$

Definition 8.9 Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible plane curve singularity. For all $s \geq 1$ we define

$$\mathfrak{M}^s(f, \tfrac{\partial}{\partial t}) := \bigoplus_{\vec{v} \in \mathcal{S}(\frac{\partial}{\partial t})} \left\{ \bigoplus_{\vec{w} \in \mathcal{M}(\frac{\partial}{\partial t})} \left(J_{\vec{v},\vec{w}} / J_{\vec{v},\vec{w}}^{s+1}\right)^* \right\} \tag{8.1}$$

and we call the direct limit

$$\mathfrak{M}(f, \tfrac{\partial}{\partial t}) := \varinjlim_{s} \mathfrak{M}^s(f, \tfrac{\partial}{\partial t})$$

the mixed Hodge structure on the nearby fundamental group of the irreducible plane curve singularity f.

Note that μ_d acts on $\mathfrak{M}(f, \frac{\partial}{\partial t})$. More precisely, $\zeta_d^{\nu} = e^{\frac{2\pi i}{d}\nu} \in \mu_d$ can be considered as map:

$$\zeta_d^{\nu} : \bigoplus_{\vec{w} \in \mathcal{M}(\frac{\partial}{\partial t})} \left(J_{\vec{v},\vec{w}} / J_{\vec{v},\vec{w}}^{s+1}\right)^* \to \bigoplus_{\vec{w} \in \mathcal{M}(\frac{\partial}{\partial t})} (J_{\zeta_d^{\nu}\vec{v},\vec{w}} / J_{\zeta_d^{\nu}\vec{v},\vec{w}}^{s+1})^*.$$

Let $\mathcal{T}$ be the monodromy of the local system

$$\bigoplus_{\vec{v} \in \mathcal{S}(\frac{\partial}{\partial t})} \left\{ \bigoplus_{\vec{w} \in \mathcal{M}(\frac{\partial}{\partial t})} \left(J_{\vec{v},\vec{w}} / J_{\vec{v},\vec{w}}^{s+1}\right)_{\mathbb{Z}}^* \right\}.$$

Define $L := (\frac{1}{d}N - \frac{1}{mk}M) : H^0\bar{B}(A^\bullet, a) \to H^0\bar{B}(A^\bullet, a)$ and moreover

$$\mathcal{L} := \bigoplus_{\vec{v},\vec{w}} L : \bigoplus_{\vec{v},\vec{w}} H^0\bar{B}(A^\bullet, a) \to \bigoplus_{\vec{v},\vec{w}} H^0\bar{B}(A^\bullet, a).$$

Then by Theorem 7.23 holds for $\vartheta \in \mathbb{C}^*$:

$$\mathcal{T} = e^{-2\pi i\mathcal{L}} \quad \text{and} \quad \mathfrak{M}^s(f, \vartheta\tfrac{\partial}{\partial t})_{\mathbf{Z}} = \vartheta^{-\mathcal{L}}\mathfrak{M}^s(f, \tfrac{\partial}{\partial t})_{\mathbf{Z}},$$

from where we may conclude that the family of MHSs

$$\{\mathfrak{M}(f, \tfrac{\partial}{\partial t})\}_{\frac{\partial}{\partial t} \in T_0\mathbf{C}}$$

is a nilpotent orbit of MHS.

Note that on the components of the direct sum (8.1) the monodromy $\mathcal{T}$ is given by

$$e^{-2\pi iL} : \left(J_{\vec{v},\vec{w}}/J^{s+1}_{\vec{v},\vec{w}}\right)^*_{\mathbf{Z}} \to \left(J_{\zeta_d\vec{v},\zeta_{mk}\vec{w}}/J^{s+1}_{\zeta_d\vec{v},\zeta_{mk}\vec{w}}\right)^*_{\mathbf{Z}},$$

where we consider the lattices on the left and on the right as sub-lattices of $H^0\bar{B}(A^\bullet, a)$.

8.4 An Example

The only purpose of the example we give in this section is to indicate that *the mixed Hodge structure on the nearby fundamental group of an irreducible plane curve singularity* detects moduli that are hidden for the MHS on the vanishing cohomology. We focus especially on effects coming from the monodromy on $\vec{J}/\vec{J}^4$. But in general, we also expect interesting information from higher periods – even on $\vec{J}/\vec{J}^3$. A systematic investigation of the invariant on classes of singularities still remains to be done.

Here we investigate the example (0.1) that was mentioned in the introduction:

$$f_\lambda(x, y) = \left(y^2 - x^3\right)^2 - \left(4\lambda x^5 y + \lambda^2 x^7\right), \quad \lambda \neq 0.$$

All f_λ have Milnor number 16 and the resolution of f_λ is shown in Figure 8.1

From this resolution and from the Milnor number, it is not difficult (cf.[BK86]) to derive that the divisor with normal crossings $D \subset Z$ has the shape that is illustrated in Figure 8.2. Moreover we see $d = 156$. Notice also that $m = 4$ and $k = 6$.

Observe that the automorphism of $(\mathbb{C}^2, 0)$ that is given by

$$\phi : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \vartheta^2 x \\ \vartheta^3 y \end{pmatrix} \tag{8.2}$$

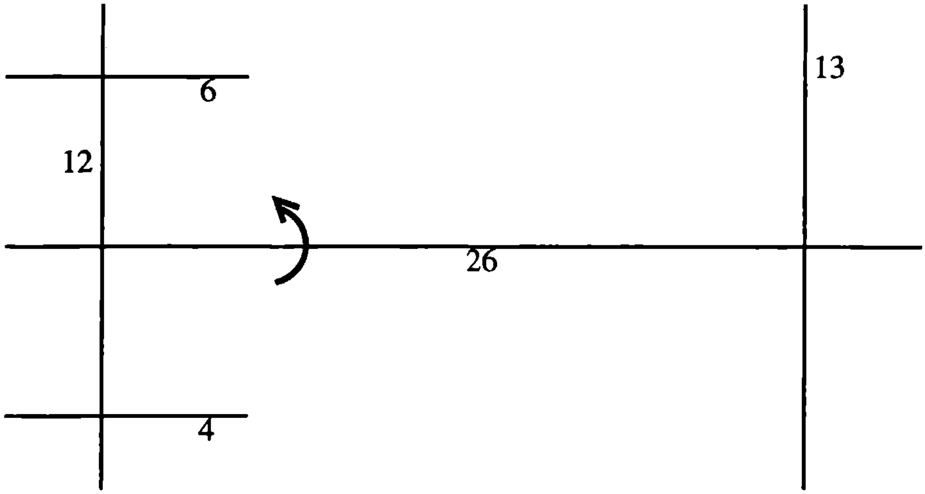


Figure 8.1: The resolution of $f_\lambda(x,y) = \left(y^2 - x^3\right)^2 - \left(4\lambda x^5 y + \lambda^2 x^7\right)$ with multiplicities e_i.

has the property that: $f_\lambda(\tilde{x},\tilde{y}) = \vartheta^{12} f_{\vartheta\lambda}(x,y)$. Or likewise, ϕ yields a commutative diagram:

$$\begin{array}{ccc} (\mathbb{C}^2,0) & \xrightarrow{\phi} & (\mathbb{C}^2,0) \\ {\scriptstyle f_{\vartheta\lambda}}\downarrow & & \downarrow{\scriptstyle f_\lambda} \\ (\mathbb{C},0) & \xrightarrow{\cdot\vartheta^{12}} & (\mathbb{C},0). \end{array}$$

Hence, the MHS on the nearby fundamental group of the pair $(f_{\vartheta\lambda}, \frac{\partial}{\partial t})$ is isomorphic to the one of $(f_\lambda, \vartheta^{12}\frac{\partial}{\partial t})$, i. e.

$$\mathfrak{M}(f_{\vartheta\lambda}, \tfrac{\partial}{\partial t}) \cong \mathfrak{M}(f_\lambda, \vartheta^{12}\tfrac{\partial}{\partial t}).$$

Therefore, we fix one f_λ and consider how $\mathfrak{M}(f_\lambda, \vartheta^{12}\frac{\partial}{\partial t})$ varies, when we change $\vartheta^{12} =: \theta$. We know that for any $s \geq 1$, the variation of MHS

$$\{\mathfrak{M}^s(f_\lambda, \theta\tfrac{\partial}{\partial t})\}_{\theta\in\mathbb{C}^*}$$

is the nilpotent orbit of MHS associated to the pair $(\mathfrak{M}^s(f_\lambda, \theta\frac{\partial}{\partial t}), \mathcal{L})$, where $\mathcal{L} = \bigoplus_{\vec{v},\vec{w}} L$ is the nilpotent endomorphism given by $L = \frac{1}{156}N - \frac{1}{24}M$.

Denote by $h_\lambda : (Z_\lambda, D_\lambda^+) \to (\Delta, 0)$ the map that is obtained by semistable reduction from f_λ. By Theorem 8.3 we have a commutative diagram

$$\begin{array}{ccc} (Z_\lambda, D_\lambda^+) & \longrightarrow & (\mathbb{C},0) \\ {\scriptstyle h_\lambda}\downarrow & & \downarrow{\scriptstyle f_\lambda} \\ (\Delta,0) & \longrightarrow & (\mathbb{C},0) \\ t & \longmapsto & t^{156}. \end{array}$$

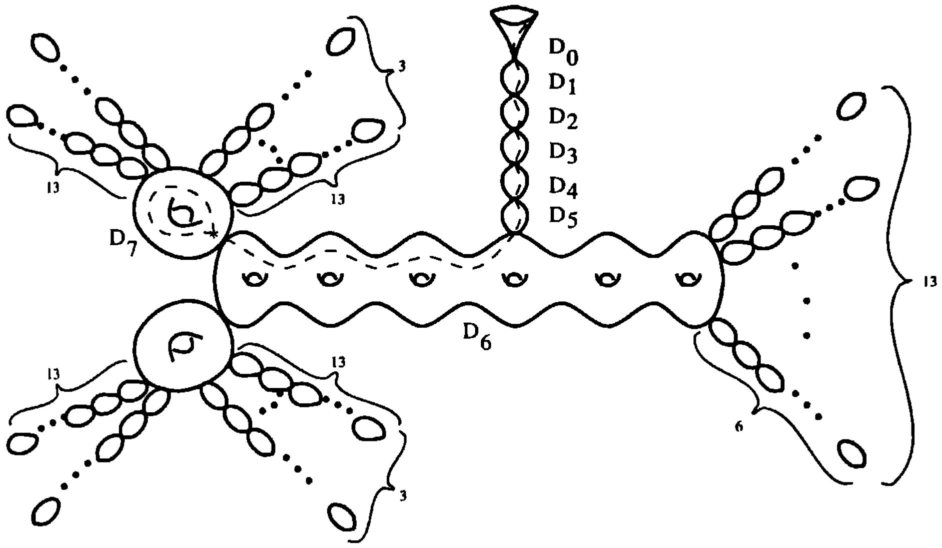


Figure 8.2: The divisor D for $f_\lambda(x,y) = \left(y^2 - x^3\right)^2 - \left(4\lambda x^5 y + \lambda^2 x^7\right)$.

To h_λ we associate the dga $A^\bullet$ and note by Corollary 6.20 on page 106 that $H^1(A^\bullet)$ is a pure Hodge structure of weight 1. Since $(\vec{J}/\vec{J}^3)^*$ has weights 1 and 2 and since N and M both lower the weights by 2, we see that N and M are both zero on $(\vec{J}/\vec{J}^3)^*$. This shows that the nilpotent orbit of MHS

$$\left\{\mathfrak{M}^2(f_\lambda, \theta\tfrac{\partial}{\partial t})\right\}_{\theta\in\mathbf{C}^*}$$

is constant. Our goal now is to show:

Proposition 8.10 *The nilpotent orbit of MHS*

$$\left\{\mathfrak{M}^3(f_\lambda, \theta\tfrac{\partial}{\partial t})\right\}_{\theta\in\mathbf{C}^*}$$

is not constant.

Proof: In order to prove this proposition we just have to show that $L \neq 0$. Choose $\vec{v} \in \mathcal{S}(\frac{\partial}{\partial t})$ and $\vec{w} \in \mathcal{M}(\frac{\partial}{\partial t})$. On the elliptic curve D_7, choose a point $* \in D_7$ and a closed path α based at $*$. Let $\omega^{(7)}$ be a holomorphic 1-form on D_7 such that

$$\int_\alpha \omega^{(7)} = 1.$$

There exists a $\mu^{(7)} \in E^1(D_7 \log p_{67})$ such that

$$\omega^{(7)} \wedge \bar{\omega}^{(7)} + d\mu^{(7)} = 0$$

and $\rho := \mathrm{Res}_{p_{67}} \mu^{(7)} \neq 0$. Now let $\eta_1^{(7)}, \dots, \eta_6^{(7)}$ be meromorphic forms on $D_1, \dots, D_6$ respectively and $\eta_0^{(7)} = \rho_{01} \frac{dp}{p} \in \bigwedge^1(\frac{dp}{p})$ such that:

$$\rho_{\imath,\imath+1} = \mathrm{Res}_{p_{\imath,\imath+1}} \eta_\imath^{(7)} = -\mathrm{Res}_{p_{\imath-1,\imath}} \eta_\imath^{(7)} = -\rho.$$

Now consider the Chen-closed element in $(A^1 \otimes A^1) \oplus A^1$ that is given by:

$$\Omega = \omega^{(7)} \otimes \omega^{(7)} \otimes \bar{\omega}^{(7)} + \omega^{(7)} \otimes \left(\mu^{(7)} + \eta_6^{(7)} + \cdots + \eta_0^{(7)} + \rho_{67}\Theta + \cdots + \rho_{01}\Theta\right).$$

And note that $N(\Omega) = \omega^{(7)} \otimes (\rho_{67} d\xi + \cdots + \rho_{01} d\xi)$. There is an $h \in A^0$ such that $h(p_{01}) = 0$ and $dh = \rho_{67} d\xi + \cdots + \rho_{01} d\xi$. It is easy to see that for this h holds

$$[N(\Omega)] = \left(\omega^{(7)} \middle| (\rho_{67} d\xi + \cdots + \rho_{01} d\xi)\right) = (-h\omega^{(7)}) = (7\rho\omega^{(7)}) \in H^0 \bar{B}_3(A^\bullet, a).$$

Similarly we see $[M(\Omega)] = (\rho\, \omega^{(7)}) \in H^0 \bar{B}_3(A^\bullet, a)$. From this we derive:

$$\begin{aligned} [L(\Omega)] &= \frac{1}{156}[N(\Omega)] - \frac{1}{24}[M(\Omega)] \\ &= (-\frac{32}{156}\rho\, \omega^{(7)}) \neq 0 \end{aligned}$$

□

Remark 8.11 Since $\mathfrak{M}^2(f_\lambda, \theta \frac{\partial}{\partial t}) = \mathfrak{M}^2(f_\lambda, \frac{\partial}{\partial t})$ for all $\theta \in \mathbb{C}^*$, we may consider the map

$$\begin{array}{ccc} \mathbb{C}^* & \to & \mathrm{Ext}_{\mathrm{MHS}}\left(\bigoplus_{\vec{v},\vec{w}} H^1(A^\bullet);\ \mathfrak{M}^2(f_\lambda, \frac{\partial}{\partial t})\right) \\ \theta & \mapsto & \left\{0 \to \mathfrak{M}^2(f_\lambda, \frac{\partial}{\partial t}) \to \mathfrak{M}^3(f_\lambda, \theta\frac{\partial}{\partial t}) \to \bigoplus_{\vec{v},\vec{w}} H^1(A^\bullet)^{\otimes 3}\right\}. \end{array} \tag{8.3}$$

The fact that $[\omega^{(7)}] \otimes [\omega^{(7)}] \otimes [\bar{\omega}^{(7)}] \in F^2(H^1(A^\bullet)^{\otimes 3})$ and $L(\Omega) \notin F^2 H^1(A^\bullet) = 0$ can be used to prove that even the map (8.3) is not constant.

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Samenvatting in het Nederlands

Over De Rham-homotopietheorie voor vlakke algebraïsche krommen en hun singulariteiten

In dit proefschrift beschouwen we vlakke complexe algebraïsche krommen; in het bijzonder onderzoeken we singulariteiten van zulke krommen lokaal. Het voornaamste doel dat we nastreven is de constructie van een invariant voor vlakke krommensingulariteiten in het kader van De Rham-homotopietheorie.

Voor compacte Riemannoppervlakken met basispunt hebben R. Hain en M. Pulte een opmerkelijke Torellistelling voor de gemengde Hodgestructuur op de fundamentaalgroep bewezen (zie Stelling 4.3 op blz. 58). Dat was een stimulans om analoga van deze stelling zowel voor niet-compacte Riemannoppervlakken als voor krommensingulariteiten te vinden.

In het eerste gedeelte van dit proefschrift beschouwen we het complement van een punt in een compact Riemannoppervlak met basispunt. Dit noemen we ook wel een *gepuncteerd Riemannoppervlak met basispunt*. We onderzoeken welke en hoeveel meer informatie de gemengde Hodgestructuur op de fundamentaalgroep van een gepuncteerd Riemannoppervlak bevat in vergelijking met een compact Riemannoppervlak.

In Hoofdstuk 1 introduceren we de door Chen ontwikkelde theorie van geïtereerde integralen, die algemeen geldt op gladde variëteiten. Daarna zijn Hoofdstukken 2 tot en met 4 gewijd aan het speciale geval van (gepuncteerde) Riemannoppervlakken met basispunt. Hoofdstuk 2 beschrijft voor (gepuncteerde) Riemannoppervlakken de constructie van de *gemengde Hodgestructuur op de fundamentaalgroep* zoals die door Hain in algemenere context werd ingevoerd. In Hoofdstuk 3 beschouwen we de extensie van het stuk van gewicht 2 door het stuk van gewicht 1 van de gemengde Hodgestructuur op de fundamentaalgroep voor zowel een gepuncteerd Riemannoppervlak met basispunt als een compact Riemannoppervlak met basispunt. Met name kijken we hoe deze invariant in de afzonderlijke gevallen verschilt. Hier speelt de canonieke divisor van dat compacte Riemannoppervlak een belangrijke rol. Hoofdstuk 4 geeft een Torellistelling voor gepuncteerde Riemannoppervlakken met basispunt analoog met de Torellistelling van Hain en Pulte.

Het tweede gedeelte van dit proefschrift gaat over irreducibele vlakke krommensingulariteiten. Om een analogon met de Torellistelling van Hain en Pulte te vinden is de eerste vereiste de *constructie* van een invariant voor zulke singulariteiten in het kader van De Rham-homotopietheorie. In dit proefschrift wordt een dergelijke constructie gegeven.

De rol van de fundamentaalgroep wordt hier overgenomen door de in Hoofdstuk 5 gedefiniëerde *nabije fundamentaalgroep*. De nabije fundamentaalgroep is een groep, die isomorf is met de fundamentaalgroep van de Milnorvezel maar wel op de centrale vezel is gedefiniëerd. De geconstrueerde structuur wordt dan ook later in Hoofdstuk 8 *de gemengde Hodgestructuur op de nabije fundamentaalgroep van de singulariteit* genoemd. In Hoofdstuk 6 introduceren we een gegradeerde differentiaalalgebra, die een essentiële rol in de constructie van de gemengde Hodgestructuur op de nabije fundamentaalgroep speelt. Verder laten we zien hoe elementen van graad 1 van deze gegradeerde differentiaalalgebra langs elementen uit de nabije fundamentaalgroep kunnen worden geïntegreerd. Hoofdstuk 7 laat zien, dat het mogelijk is om in de reeds genoemde gegradeerde differentiaalalgebra geïtereerde integralen met 1-vormen te definiëren langs elementen uit de nabije fundamentaalgroep. Na de definities in Hoofdstuk 6 heeft de formule voor geïtereerde integralen een voor de hand liggende gedaante, maar het is niet eenvoudig te bewijzen dat de in deze formule benodigde limieten bestaan. Deze manier om op de centrale vezel te integreren lijkt nieuw, zelfs al voor gewone integralen.

In Hoofdstuk 8 laten we zien hoe de constructies uit de Hoofdstukken 5 tot en met 7 voor irreducibele vlakke krommensingulariteiten kunnen worden toegepast. Uiteindelijk wordt het proefschrift afgerond met een voorbeeld, waaruit blijkt dat de gemengde Hodgestructuur op de nabije fundamentaalgroep in staat is moduli op te sporen, die onzichtbaar zijn voor de gemengde Hodgestructuur op de verdwijnende cohomologie.

Zusammenfassung in deutscher Sprache

Über De Rham-Homotopietheorie für ebene algebraische Kurven und deren Singularitäten

In dieser Doktorarbeit betrachten wir ebene algebraische Kurven; insbesondere untersuchen wir Singularitäten solcher Kurven lokal. Ziel ist die Konstruktion einer De Rham-Homotopie-theoretischen Invarianten für ebene Kurvensingularitäten.

Bei kompakten Riemannschen Flächen mit Basispunkt haben R. Hain und M. Pulte einen bemerkenswerten Torellisatz für die gemischte Hodgestruktur auf der Fundamentalgruppe bewiesen (siehe Satz 4.3 auf Seite 58). Dies war eine Anregung, ähnliche Aussagen für sowohl nicht-kompakte Riemannsche Flächen als auch für Kurvensingularitäten zu finden.

Im ersten Teil dieser Dissertation betrachten wir das Komplement eines Punktes innerhalb einer kompakten Riemanschen Fläche mit Basispunkt. Dies nennen wir eine *punktierte Riemannsche Fläche mit Basispunkt.* Wir untersuchen, welche Information und wieviel mehr Information die gemischte Hodgestruktur auf der Fundamentalgruppe einer punktierten Riemannschen Fläche im Gegensatz zu einer kompakten Riemannschen Fläche enthält.

In Kapitel 1 stellen wir die von Chen entwickelte Theorie iterierter Integrale vor, welche allgemein auf Mannigfaltigkeiten gilt. Kapitel 2 bis einschließlich 4 sind dann ausschließlich dem Fall (punktierter) Riemannscher Flächen mit Basispunkt gewidmet. Kapitel 2 beschreibt für (punktierte) Riemannsche Flächen die Konstruktion der gemischten Hodgestruktur auf der Fundamentalgruppe, so wie sie von Hain allgemein für komplexe algebraische Varietäten eingeführt wurde. In Kapitel 3 betrachten wir die Erweiterung des Teils der gemischten Hodgestruktur auf der Fundamentalgruppe von Gewicht 2 durch den Teil von Gewicht 1 für sowohl punktierte Riemannsche Flächen mit Basispunkt als auch kompakte Riemannsche Flächen mit Basispunkt. Insbesondere untersuchen wir, wie diese Invariante sich in den jeweiligen Fällen unterscheidet.

Der zweite Teil dieser Arbeit handelt von irreduzibelen ebenen Kurvensingularitäten. Ein Analogon zum Torellisatz von Hain und Pulte zu finden, erfordert hier zuallererst die *Konstruktion* einer De Rham-Homotopie-theoretischen Invarianten für solche Singularitäten. In dieser Doktorarbeit wird eine derartige Konstruktion durchgeführt.

Die Rolle der Fundamentalgruppe wird hier durch die in Kapitel 5 definierte *nahe Fundamentalgruppe* übernommen. Die nahe Fundamentalgruppe ist isomorph zur Fundamentalgruppe der Milnorfaser aber ist dennoch auf der zentralen Faser definiert. Die konstruierte Struktur wird dann auch später in Kapitel 8 *die gemischte Hodgestruktur auf der nahen Fundamentalgruppe der Singularität* genannt. In Kapitel 6 führen wir eine graduierte Differentialalgebra ein, die eine wesentliche Rolle bei der Konstruktion der gemischten Hodgestruktur auf der nahen Fundamentalgruppe spielt. Weiter zeigen wir, wie Elemente von Grad 1 dieser graduierten Differentialalgebra entlang von Elementen aus der nahen Fundamentalgruppe integriert werden können. Kapitel 7 zeigt, daß es möglich ist, in der oben genannten graduierten Differentialalgebra iterierte Integrale von 1-Formen längs Elementen aus der nahen Fundamentalgruppe zu definieren. Nach den Definitionen in Kapitel 6 liegt die Gestalt der Formel für iterierte Integrale auf der Hand. Trotzdem ist es nicht einfach zu beweisen, daß die in der Definition benötigten Grenzwerte existieren. Auf solche Art und Weise auf der zentralen Faser zu integrieren, scheint selbst für gewöhnliche Linienintegrale neu zu sein.

In Kapitel 8 stellen wir dar, wie die Konstruktionen der Kapitel 5 bis 7 Verwendung finden für irreduzibele ebene Kurvensingularitäten. Wir schließen die Doktorarbeit mit einem Beispiel ab. Dieses Beispiel zeigt, daß die gemischte Hodgestruktur auf der nahen Fundamentalgruppe in der Lage ist, Moduli aufzuspüren, die für die gemischte Hodgestruktur auf der verschwindenden Kohomologie verborgen sind.

Curriculum vitae

Ik ben op 15 mei 1966 te Straelen in Duitsland geboren. In het nederrijnse dorpje Twisteden leefde ik tot mijn negentiende. Aldaar ging ik van 1972 tot 1976 naar de *Grundschule*. Vervolgens doorliep ik van 1976 tot 1985 het Kardinal-von-Galen-Gymnasium in de bedevaartsplaats Kevelaer. Van oktober 1985 tot en met december 1986 moest ik in militaire dienst.[1]

Nog in hetzelfde jaar 1986 ging ik wiskunde met bijvak natuurkunde studeren op de Friedrich-Wilhelms-Universität te Bonn. Drie jaar lang, van 1989 tot en met 1992, ben ik studentassistent aan de universiteit geweest. In 1990 heb ik zes weken als *Werkstudent* in een onderzoeksafdeling bij Siemens in München gewerkt. Twee jaar heb ik onder begeleiding van Prof. Dr. E. Brieskorn aan mijn Diplomarbeit gewerkt. Hier zijn de twee publicaties: [BK96] en [Kae96] uit voortgekomen. In oktober 1992 ben ik geslaagd voor het *Diplom* met de aantekening "Ausgezeichnet". Voor een deel van het in mijn *Diplomarbeit* verrichte onderzoek werd in 1993 de *Felix-Hausdorff-Gedächtnispreis der Universität Bonn* aan mij uitgereikt. Ik werkte nog een half jaar als *Wissenschaftlicher Mitarbeiter* aan de Universiteit Bonn, voordat ik begin mei 1993 met een tweejarige beurs van de EEG in het kader van het *Human Capital and Mobility programme* naar de Katholieke Universiteit Nijmegen ging om onder leiding van Prof. Dr. J.H.M. Steenbrink promotieonderzoek te verrichten. Voordat ik in juni 1995 als twAiO[2] het promotieonderzoek voortzette, heb ik mei 1995 als *Maître de Conférences invité* aan de Université d'Angers in Frankrijk doorgebracht. In het voorjaar van 1996 bracht ik een werkbezoek van drie maanden aan Prof. Dr. R. M. Hain van Duke University, NC in de Verenigde Staten. Dit bezoek werd mogelijk gemaakt door de steun van de NWO en de Universiteit Nijmegen. Vanaf juni 1997 tot eind september 1997 was ik *junior onderzoeker* aan de Universiteit Nijmegen. Met mijn promotie zit mijn tijd in Nijmegen er voorlopig op.

Vanaf oktober 1997 ga ik voor een jaar als post-doc aan de Universiteit te Utrecht werken in het kader van het door de hoogleraren Dijkgraaf, Faber, Van der Geer, Looijenga en Oort geleide NWO-projekt "Algebraic Curves and Riemann Surfaces: geometry, arithmetic and applications".

Sinds ik in Nederland werk, ben ik drie keer *Zwarte Piet* geweest.

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[1] In 1991 werd ik *Kriegsdienstverweigerer*.

[2] Dit zijn de laatste twee jaar van een AiO-schap.

Stellingen

behorende bij het proefschrift
On De Rham Homotopy Theory of Plane Algebraic Curves and their Singularities
van Rainer Helmut Kaenders

1. **Stelling (E. Selling)** Zij M een vrij $\mathbb{Z}$-moduul van rang $r-1$ voorzien van een symmetrische bilineaire vorm

$$\beta : M \times M \longrightarrow \mathbb{R}.$$

Stel dat er elementen $\theta_1, \ldots, \theta_r \in M$ bestaan, zodat:

(a)	$\beta(\theta_i, \theta_j) \quad < \quad 0$	voor ieder paar $i \neq j$ uit $\{1, \ldots, r\}$,
(b)	$\theta_1 + \cdots + \theta_r = 0$,	
(c)	$\theta_1, \ldots, \widehat{\theta_i}, \ldots, \theta_r$	is een basis van M voor iedere $i = 1, \ldots, r$.

Dan is de verzameling $\{\theta_1, \ldots, \theta_r\}$, op een gemeenschappelijk teken van al zijn elementen na, uniek bepaald.
(zie. Eduard Selling, Würzburg. *Ueber die binären und ternären quadratischen Formen.* J. Reine Angew. Math., 77(Heft 2 und 3):143–229, 1874.)

2. De Seifertvorm van een reducibele vlakke krommensingulariteit bepaalt de onderlinge snijgetallen van de takken (zie [Kae96]).

3. Een $A\Gamma$-diagram van een (reducibele) vlakke krommensingulariteit legt het topologische type volledig vast (zie [BK96]).

4. Sometimes, Hodge theory is hodgepodge theory.

5. Das Volumen eines Körpers, der durch Drehung eines ebenen Gebietes um eine dieses Gebiet nicht schneidende Achse entsteht, ist gleich dem Produkt aus dem Flächeninhalt dieses Gebietes mit dem Umfang des Kreises, den der Schwerpunkt dieses Gebietes bei der Drehung beschreibt (Zweite Guldinsche Regel).

6. Voor drie vectoren a, b, $c \in \mathbb{R}^3$ geldt: $\boxed{a \times (b \times c) = b\langle a, c\rangle - c\langle a, b\rangle}$. Deze regel staat ook bekend onder de naam BACCAB (spreek uit: "Baktsap").

7. Ein *Gebiet* in $\mathbb{C}$ ist eine nichtleere zusammenhängende offene Teilmenge von $\mathbb{C}$. Dann gilt: Ein Gebiet G ist einfach zusammenhängend genau dann wenn es sich als endliche oder abzählbar unendliche Vereinigung $\bigcup_{j=1}^{p \text{ oder } \infty} G_j$ von sternförmigen Gebieten $\{G_j\}_{j=1,\ldots,\infty}$ schreiben läßt, derart, daß für jedes $N \in \{1, \ldots, p \text{ oder } \infty\}$ gilt:

$$\left(\bigcup_{j=1}^{N} G_j\right) \cap G_{N+1} \text{ ist ein Gebiet.}$$

8. De incarnaties van de platonische idee van een *torus* zijn: in Duitsland een *Adventskranz*, in de VS een *donut* en in Nederland een *fietsband.*

9. Unter allen mathematischen Disziplinen ist die Topologie die sinnlichste.

10. Nog steeds is de echte wiskunde *unplugged.*

11. Het stuk *'Het ruikt hier naar gas'* van Godfried Bomans in de bundel *'Van de hak op de tak'*, (Elsevier Amsterdam/Brussel, 1965) is in staat zelfs bij een welwillende Duitser gevoelens van verontwaardiging op te wekken.

12. Er bestaat een in Duitsland nogal veel aangetroffen vooroordeel dat er een vergeetfunctor van de Duitse naar de Nederlandse taal zou zijn.

13. Er valt voor te pleiten het vak *biologie* tot *leefkunde* te herbenoemen.